Parametric Signal Modeling

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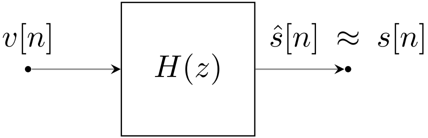
Last lecture

- The PSD is the DTFT of the autocorrelation function
- The PSD may be two-sided or one-sided. Careful with conventions!
- The periodogram method estimates the PSD directly from the magnitude squared of the DFT of the windowed signal
- The periodogram is an biased estimator of the PSD, and it has large variance. Hence, the periodogram must be averaged to produce useful estimates
- The Welch method averages several periodograms
- The Welch method breaks the data into overalapping segments, each of length L. Usually, the segments overlap by $L/2$.
- Differently from the periodogram and Welch method, the Blackman-Tukey method estimates the PSD by computing the DFT of the estimated autocorrelation function
- Although the estimator of the autocorrelation function may be unbiased, the PSD estimate is biased. Windows with non-negative frequency response are typically preferred e.g., Bartlett
- Increasing the sequence length Q improves accuracy. Reducing the window length L improves accuracy at the expense of poorer frequency resolution.

Parametric signal modeling

- Another representation of signals
- We'll model *complicated* signals as the output of some system to white noise or to an impulse.
- Hence, the signal will be described by the parameters (coefficients) of the system
- We'll cover the **all-pole model** or **autoregressive (AR) model**

All-pole model



Given a signal $s[n]$, find $H(z)$ and $v[n]$ such that

$$s[n] \approx \hat{s}[n] = h[n] * v[n]$$

- The **all-pole model** assumes

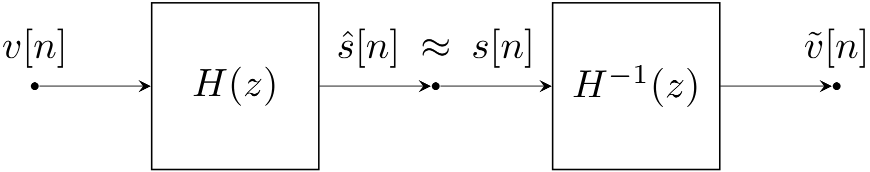
$$H(z) = \frac{G}{1 - a_1 z^{-1} - \ldots - a_N z^{-N}}$$

  all zeros are at the origin
- If $s[n]$ is a finite-energy deterministic signal, $v[n]$ is chosen to be an unit impulse
- If $s[n]$ is a WSS random process, $v[n]$ is chosen to be a white noise process with unit average power

Question: how to find the coefficients $a_1, \ldots, a_N$?

All-pole model

Consider the inverse system



The inverse is an FIR system: $H^{-1}(z) = \frac{1}{G}(1 - a_1 - \ldots - a_N z^{-N})$. Therefore,

$$Gv[n] \approx s[n] - \sum_{k=1}^{N} a_k s[n-k]$$

Defining the error

$$e_m[n] = Gv[n] - \left(s[n] - \sum_{k=1}^{N} a_k s[n-k]\right) \qquad \text{(modeling error)}$$

We want to find coefficients $a_1, \ldots, a_n$ that minimize the **mean-square error**

$$\langle |e_m[n]|^2 \rangle = \begin{cases} \sum_{n=0}^{L} |e_m[n]|^2, & s[n] \text{ deterministic} \\ \mathbb{E}\left(|e_m[n]|^2\right), & s[n] \text{ random} \end{cases}$$

From the properties of the mean-square error, it can be shown that the error is minimum when

$$\langle s[n-i] e_m[n] \rangle = 0, \quad i = 0, \ldots, N$$

The error is orthogonal to all inputs. This is known as the **orthogonality principle**

Applying the orthogonality principle:

$$\left\langle s[n-i]\Big(Gv[n]-(s[n]-\sum_{k=1}^{N}a_k s[n-k])\Big)\right\rangle = 0, \quad i=0,\ldots,L$$

We'll only consider the cases when $i=1,\ldots,N$:

$$\left\langle s[n-i]\Big(Gv[n]-(s[n]-\sum_{k=1}^{N}a_k s[n-k])\Big)\right\rangle = 0, \quad i=1,\ldots,N$$

$$\langle s[n-i]s[n]\rangle - \sum_{k=1}^{N}a_k\langle s[n-i]s[n-k]\rangle) = 0, \quad i=1,\ldots,N$$

(from causality $\langle s[n-i]v[n]\rangle = 0$)

Note that $\langle s[l]s[m]\rangle$ is equivalent to either the **deterministic autocorrelation function** if $s[n]$ is deterministic, or $\langle s[l]s[m]\rangle$ is equivalent to the **autocorrelation function** if $s[n]$ is random

$$\langle s[l]s[m]\rangle = \begin{cases} c_{ss}[l-m], & s[n] \text{ deterministic} \\ \phi_{ss}[l-m], & s[n] \text{ random} \end{cases}$$

If $s[n]$ is deterministic:

$$c_{ss}[i] = \sum_{k=1}^{N} a_k c_{ss}[k-i], \quad i = 1, \ldots, N$$

If $s[n]$ is random:

$$\phi_{ss}[i] = \sum_{k=1}^{N} a_k \phi_{ss}[k-i], \quad i = 1, \ldots, N$$

These equations are known as **normal equations** or **Yule-Walker equations**.

They form a system of N linear equations of N variables. Hence, there is at most one solution.

In matrix notation

Define

$$r[i] = \begin{cases} c_{ss}[i], & s[n] \text{ deterministic} \\ \phi_{ss}[i], & s[n] \text{ random} \end{cases}$$

We can write the **normal equations** in matrix notation:

$$Ra = r$$

$$\begin{bmatrix} r[0] & r[1] & \dots & r[N-1] \\ r[1] & r[0] & \dots & r[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ r[N-1] & r[N-2] & \dots & r[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} r[1] \\ r[2] \\ \vdots \\ r[N] \end{bmatrix}$$

R is the **autocorrelation matrix**. Note that R is symmetric $R = R^T$.

In Matlab: `>> a = R\r`

Determining the gain

Once the coefficients $a_1, \ldots, a_N$ have been found, we just need to determine the gain G to completely specify $H(z)$.

We can pick G so that the mean-square value of the model output matches the mean-square value of the signal $s[n]$:

$$\langle \hat{s}^2[n] \rangle = \langle s^2[n] \rangle$$

$$G^2 = r_{ss}[0] - \sum_{k=1}^{N} a_k r_{ss}[k] \qquad \text{(after some algebra)}$$

Spectrum analysis with all-pole model

Deterministic case:

$$|S(e^{j\omega})|^2 \approx |H(e^{j\omega})|^2 |V(e^{j\omega})|^2 = |H(e^{j\omega})|^2$$

since by assumption $v[n] = \delta[n]$, it follows that $|V(e^{j\omega})|^2 = 1$

Random case:

$$P_{ss}(e^{j\omega}) \approx |H(e^{j\omega})|^2 P_{vv}(e^{j\omega}) = |H(e^{j\omega})|^2$$

since by assumption $v[n]$ is white noise, it follows that $P_{vv}(e^{j\omega}) = 1$

Conclusion:

$$|H(e^{j\omega})|^2 = \left| \frac{G}{1 - \sum_{k=1}^{N} a_k e^{-j\omega k}} \right|^2$$

is the spectrum estimate of $s[n]$.

All-pole analysis of speech signals

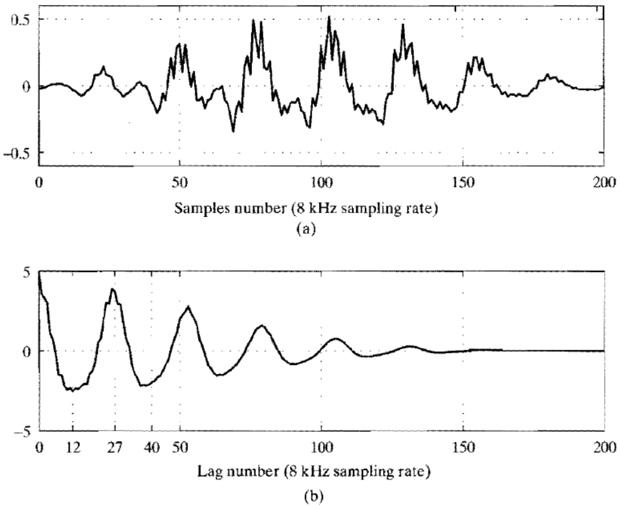


Figure 11.11 (a) Windowed voiced speech waveform. (b) Corresponding autocorrelation function (samples connected by straight lines).

All-pole analysis of speech signals

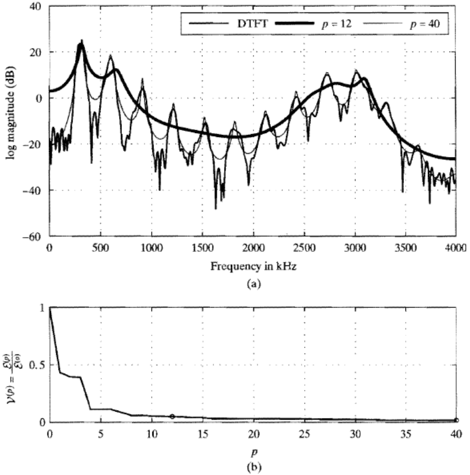
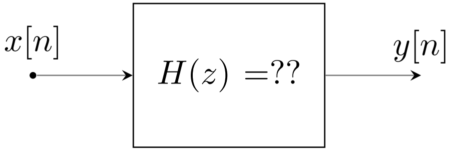


Figure 11.12 (a) Comparison of DTFT and all-pole model spectra for voiced speech segment in Figure 11.11(a). (b) Normalized prediction error as a function of *p*.

Revisiting the plant identification problem



Inject a small noise of known PSD $\Phi_{xx}(e^{j\omega})$ at the input and measure the noise PSD at the output $\Phi_{yy}(e^{j\omega})$:

$$|H(e^{j\omega})|^2 = \frac{\Phi_{yy}(e^{j\omega})}{\Phi_{xx}(e^{j\omega})} \qquad \text{(only know the magnitude)}$$

We can obtain the phase response by computing the Hilbert transform of the log-magnitude

$$\arg(H(e^{j\omega})) = -\mathcal{H}\{\ln|H(e^{j\omega})|\}$$

This procedure only works if $H(z)$ is causal and minimum phase i.e., it is stable and has a stable inverse.

Summary

- Parametric signal modeling is a form of representing complicated signals as the output to some system to an impulse or to a white noise process
- The all-pole or autoregressive model assumes that all zeros are at the origin
- Under the all-pole model the signal is described by the parameters $a_1, \ldots, a_N$, i.e., the system coefficients
- The coefficients can be determined by solving the normal equations, also known as Yule-Walker equations
- Solving these equations require estimates of either the deterministic autocorrelation function, if the signal is deterministic, or the autocorrelation function, if the system is random
- The magnitude square of the system is equivalent to the spectrum estimate of the signal