Power Spectrum Density Estimation

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Last lecture

- Leakage and resolution are important considerations in spectrum analysis
- By properly choosing windows we can minimize these issues
- Kaiser window is a nearly optimal choice. Must choose correct β and window length L
- β controls the ratio between the amplitudes of the main-lobe and the largest side-lobe i.e., β controls the amount of leakage.
- The larger the main-lobe width, the smaller the resolution
- By increasing the window length we reduce the main-lobe width and consequently improve the resolution
- Time-dependent Fourier transform or short-time Fourier transform allows us to keep track of frequency variation in time
- Spectrogram is a commonly used way to display the TDFT
- In the spectrogram the TDFT is sampled both in time and in frequency
- The window length determines the resolution of the spectrogram

Today's lecture

How to estimate the power spectrum density (PSD) of WSS signals.

Periodogram

Welch's method

Blackman-Tukey method

Revisiting stationary random signals

- Stationarity refers to time-invariance of some or all statistics of a random process
- We will limit our studies to wide-sense stationary (WSS) random processes
- The mean of a WSS process is constant, while the autocorrelation function depends only on the time deference m, and not on the absolute time n.

$$\mathbb{E}(x[n]) = \mu \qquad \text{(constant mean)}$$

$$\mathbb{E}(x[m+n]x^*[n]) = \phi_{xx}[m]$$

(autocorrelation function depends only on time difference m)

- The power spectrum density (PSD) of a WSS process is the Fourier transform of the autocorrelation function:

$$P(\omega) \equiv \Phi_{xx}(e^{j\omega}) = \mathcal{F}\{\phi_{xx}[m]\}$$

  The PSD is purely real, even symmetric, and non-negative.
- Average power

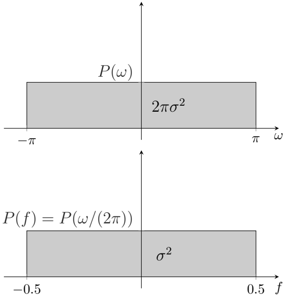
$$\mathbb{E}(|x[n]|^2) = \phi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})d\omega \qquad \text{(average power)}$$

Conventions

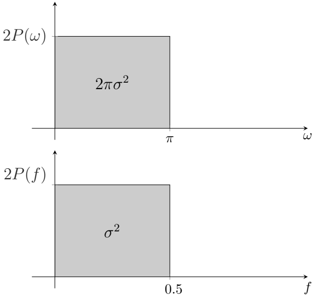
A zero-mean random process has PSD $P(\omega)$ and average power σ^2.

$$\sigma^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\omega)d\omega = \int_{-0.5}^{0.5} P(f)df$$

Two-sided PSD



One-sided PSD



Generally, one-sided PSD is used when dealing with real signals, while two-sided PSD is used when dealing with complex signals.

Matlab conventions

The Matlab functions for PSD estimation follow these conventions:

- Matlab returns **one-sided PSD** if signal is real and **two-sided PSD** if the signal is complex.
- You can force Matlab to return two-sided PSD by specifying frequency vector ω with negative frequencies
- When working with normalized frequency $f = \omega/(2\pi)$, Matlab returns

$$\texttt{Pf} = \begin{cases} 2P(f), & \text{if one-sided PSD} \\ P(f), & \text{if two-sided PSD} \end{cases}$$

- When working with normalized frequency ω, Matlab returns

$$\texttt{Pw\_norm} = \begin{cases} \frac{1}{\pi}P(\omega), & \text{if one-sided PSD} \\ \frac{1}{2\pi}P(\omega), & \text{if two-sided PSD} \end{cases}$$

Therefore, the area under the PSD in Matlab is equal to the average power, even when working with ω.

Outline

Periodogram

Welch's method

Blackman-Tukey method

The periodogram

The **periodogram** of $x[n]$ is given by

$$P(\omega) \equiv \frac{1}{LU} |\mathcal{F}\{w[n]x[n]\}|^2, \quad \text{where} \quad U = \frac{1}{L} \sum_{n=0}^{L-1} |w[n]|^2$$

The periodogram is the magnitude squared of the DTFT of $w[n]x[n]$, normalized by the window length L and the window power U.

The sampled periodogram can be computed efficiently using the DFT:

$$P[k] = P\left(\frac{2\pi k}{N}\right) = \frac{1}{LU} |\mathrm{DFT}\{w[n]x[n]\}|^2$$

For $N \geq L$, the N-point DFT of $w[n]x[n]$ is simply the DTFT $\mathcal{F}\{w[n]x[n]\}$ sampled with period $2\pi/N$.

Periodogram and the deterministic autocorrelation function

Define $v[n] = w[n]x[n] \iff V(e^{j\omega}) = \mathcal{F}\{w[n]x[n]\}$

$$\begin{aligned} P(\omega) &= \frac{1}{LU}|\mathcal{F}\{w[n]x[n]\}|^2 = \frac{1}{LU}\Big(V(e^{j\omega})V^*(e^{j\omega})\Big) \\ &= \frac{1}{LU}\mathcal{F}\{v[n] * v^*[-n]\} && (|V(e^{j\omega})|^2 = \mathcal{F}\{v[n] * v^*[-n]\}) \\ &= \frac{1}{LU}\mathcal{F}\{c_{vv}[m]\} && (\text{by definition } c_{vv}[m] = v[m] * v^*[-m]) \end{aligned}$$

The periodogram is the DTFT of the **deterministic autocorrelation function** $c_{vv}[m]$, normalized by the window length L and the window power U.

Writing $c_{vv}[m]$ in terms of $x[n]$ and $w[n]$:

$$c_{vv}[m] = (x[m]w[m]) * (x^*[-m]w^*[-m]) = \sum_{n=0}^{L-1} x[n]x^*[m-n]w[n]w^*[m-n]$$

The periodogram as a random variable

The periodogram is a function of the random variable $x[n]$, therefore it is itself a random variable

The mean of the periodogram

$$
\begin{aligned}
\mathbb{E}(P(\omega)) &= \mathbb{E}\left(\frac{1}{LU}\mathcal{F}\{c_{vv}[m]\}\right) = \mathbb{E}\left(\frac{1}{LU}\sum_{m=L-1}^{L-1} c_{vv}[m]e^{-j\omega m}\right) \\
&= \frac{1}{LU}\sum_{m=L-1}^{L-1} \mathbb{E}(c_{vv}[m])e^{-j\omega m} \\
&= \frac{1}{LU}\sum_{m=L-1}^{L-1} c_{ww}[m]\phi_{xx}[m]e^{-j\omega m} && \text{(the window is deterministic)} \\
&= \frac{1}{2\pi LU} C_{ww}(e^{j\omega}) * \Phi_{xx}(e^{j\omega}) && \text{(product in time} \Longleftrightarrow \text{convolution in frequency)}
\end{aligned}
$$

Conclusion: *in average*, the periodogram is the PSD of $x[n]$, $\Phi_{xx}(e^{j\omega})$, *smeared* by the deterministic PSD of the window $C_{ww}(e^{j\omega}) = |W(e^{j\omega})|^2$

The periodogram as a random variable

The variance of the periodogram
It can be shown that

$$\mathrm{var}(P(\omega)) \approx \Phi_{xx}^2(e^{j\omega})$$

Conclusion: the periodogram has large variance, making it useless by itself for estimating the power spectrum of a random signal.

To mitigate this problem we can average several periodograms.

$$\bar{P}(\omega) = \frac{1}{N_{avg}} \sum_{i=0}^{N_{avg}-1} P_i(\omega)$$

where $P_i(\omega) = \Phi_{xx}(e^{j\omega}) + N_i(\omega)$, and $N_i(\omega)$ is the error of the ith periodogram. If the error $N_i(\omega)$ has variance σ_N^2. Then,

$$\mathrm{var}(\bar{P}(\omega)) = \frac{1}{N_{avg}} \sigma_N^2$$

The variance is reduced by the number of averages N_{avg}

Periodogram in Matlab

Periodogram definition:

$$P[k] \equiv \frac{1}{LU}|V(e^{j2\pi/Nk})|^2, \quad \text{where} \quad U = \frac{1}{L}\sum_{n=0}^{L-1}|w[n]|^2$$

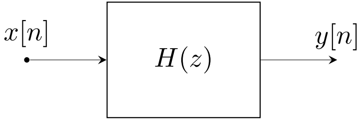
$$= \texttt{1/(L*U)*abs(fft(x.*window, N)).^2} \qquad \text{(Matlab code)}$$

This returns the two-sided periodogram at frequencies $\omega_k = 2\pi/Nk, k = 0, \dots, L-1$.
Matlab's periodogram function:

```
>> dw = 2*pi/N
>> Pw = 2*pi*periodogram(x, window, -pi:dw:pi-dw)
```

Careful with conventions: this particular call of the `periodogram` function returns a **two-sided periodogram** for frequencies in the interval $[-\pi, \pi)$. The multiplicative factor `2*pi` corrects for the PSD normalization done by Matlab.

Example



Let's assume that $H(z) = 1 - z^{-2}$. Hence

$$H(e^{j\omega}) = 1 - e^{-j2\omega} = 2j\sin(\omega)e^{-j\omega}$$

And for the power spectrum density:

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\Phi_{xx}(e^{j\omega}) = 4\sin^2(\omega)\Phi_{xx}(e^{j\omega})$$

$\Phi_{yy}(e^{j\omega})$ is the **two-sided PSD** of $y[n]$, as usual.

If $x[n]$ is a white noise with average power σ_x^2

$$\Phi_{yy}(e^{j\omega}) = 4\sin^2(\omega)\sigma_x^2$$

See Matlab script: Canvas/Files/Matlab/`periodogram_example.m`

Outline

Welch's method of estimating the PSD

Welch's PSD estimate is obtained by averaging several periodograms

1. Divide $x[n]$ into K overlapping segments

$$x_r[n] = x[rR+n], \quad n = 0, \ldots, L-1$$

   Usually $R = L/2$. i.e., the subsequences overlap by 50%.

2. Compute the periodogram for each sequence $x_r[n]$

$$P_r(\omega) = \frac{1}{LU} |\mathcal{F}\{x_r[n]w[n]\}|^2, \quad \text{where} \quad U = \frac{1}{L}\sum_{n=0}^{L-1} |w[n]|^2$$

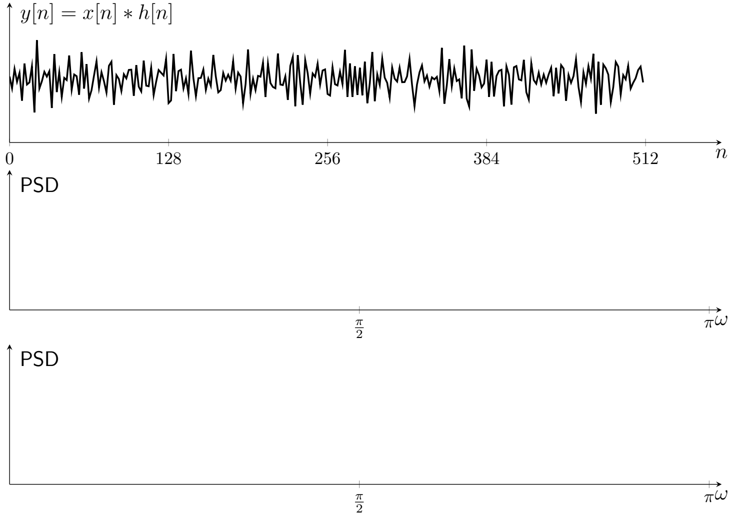
$$P_r[k] = \frac{1}{LU} |\mathrm{DFT}\{x_r[n]w[n]\}|^2$$

3. Average the periodograms $P_r[k]$:

$$\bar{P}[k] = \frac{1}{K}\sum_{r=0}^{K-1} P_r[k]$$

Welch's method example

Same scenario as before: white noise $x[n]$ filtered by $H(z) = 1 - z^{-2}$
These plots assume $R = 128$, $L = N = 256$.



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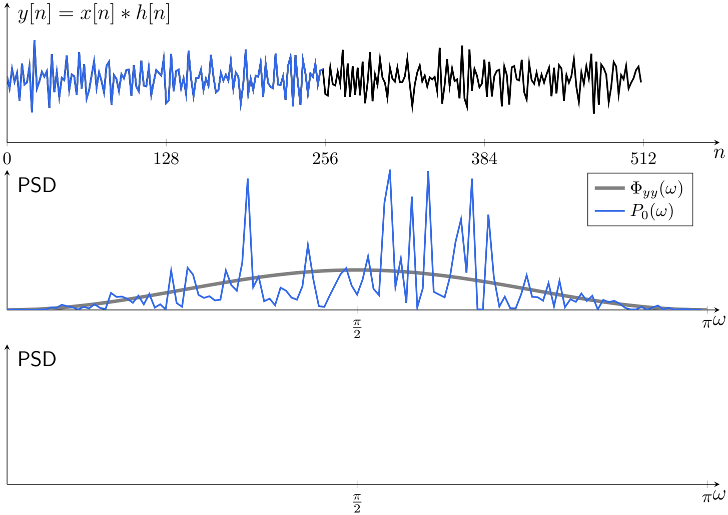

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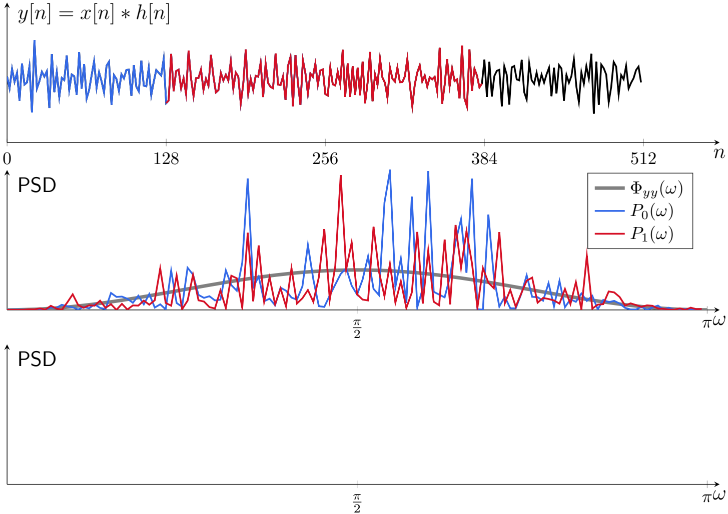

Welch's method example

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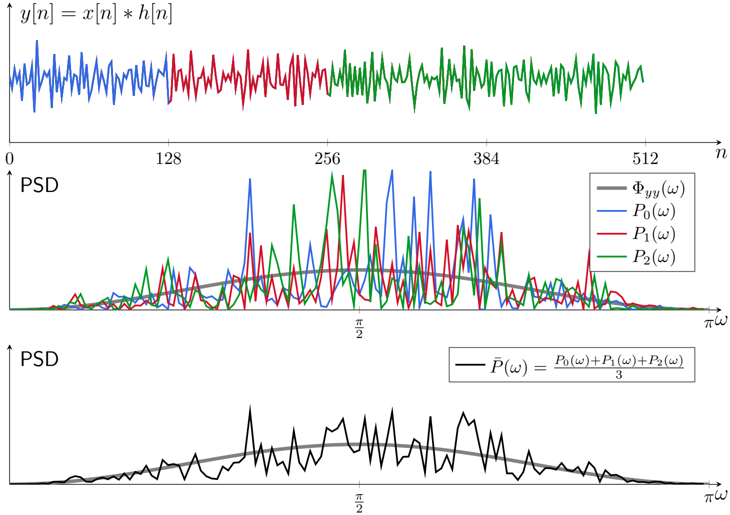

Welch's method and spectrogram

Recall that the spectrogram is defined by

$$X[r,k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km} \qquad \text{(spectrogram)}$$

For each r, the spectrogram is the DFT of $x[rR+m]w[m]$. Hence, we can calculate Welch's PSD estimate from the spectrogram

$$\bar{P}[k] = \frac{1}{LUK} \sum_{r=0}^{K-1} |X[r,k]|^2 \qquad \text{(Welch's PSD estimate)}$$

This is equivalent to averaging the squared magnitude of the spectrogram over time r.

Welch's method in Matlab

- Using the pwelch function:

```
>> dw = 2*pi/Nfft
>> Pw = 2*pi*pwelch(x, window, L-R, -pi:dw:pi-dw)
```

(two-sided PSD estimate)

The same convention considerations for the periodogram apply to the pwelch function.

- Using the spectrogram

```
>> dw = 2*pi/Nfft
>> S = spectrogram(x, window, L-R, -pi:dw:pi-dw)
>> Pw = mean(1/(L*U)*abs(S).^2, 2)
```

(two-sided PSD estimate)

L is the window length and U is the window power.

See Matlab script: Canvas/Files/Matlab/welch_example.m

Outline

Periodogram

Welch's method

Blackman-Tukey method

Blackman-Tukey PSD estimate

The Blackman-Tukey method first estimates the autocorrelation function $\hat{\phi}_{xx}[m]$. The PSD follows by computing the DFT of $\hat{\phi}_{xx}[m]$.

1. Estimate the autocorrelation function from Q samples of $x[n]$

$$\hat{\phi}_{xx}[m] = \frac{1}{Q - |m|} \sum_{n=0}^{Q-1-|m|} x[m+n]x[n], \quad |m| \leq Q - 1$$

2. Multiply autocorrelation by window $w[m]$ of length $L = 2M - 1$.

$$s[m] = \begin{cases} \hat{\phi}_{xx}[m]w[m], & -(M-1) \leq m \leq (M-1) \\ 0, & \text{otherwise} \end{cases}$$

3. Form DFT-even sequence

$$s_e[m] = s[((m))_N]$$

   This guarantees that the PSD is linear phase.

4. Compute DFT

$$S[k] = \left| \sum_{m=0}^{N-1} s[m]e^{-j(2\pi/N)k} \right|, \quad k = 0, \ldots, N - 1$$

The mean of Blackman-Tukey PSD estimate

The estimator of the autocorrelation function from Q samples of $x[n]$

$$\hat{\phi}_{xx}[m] = \frac{1}{Q-|m|} \sum_{n=0}^{Q-1-|m|} x[m+n]x[n], \quad |m| \le Q-1$$

has mean

$$\begin{aligned}\mathbb{E}(\hat{\phi}_{xx}[m]) &= \frac{1}{Q-|m|} \sum_{n=0}^{Q-1-|m|} \mathbb{E}(x[m+n]x[n]) = \frac{1}{Q-|m|} \sum_{n=0}^{Q-1-|m|} \phi_{xx}[m] \\ &= \phi_{xx}[m], \quad |m| \le L-1\end{aligned}$$

Therefore, this estimator is **unbiased** i.e., its mean is equal to the ideal value.

The mean of Blackman-Tukey PSD estimate

The mean of the PSD estimate is simply

$$
\begin{aligned}
\mathbb{E}(P(\omega)) &= \sum_{m=0}^{N-1} \mathbb{E}(\hat{\phi}_{xx}[m]w[m])e^{-j\omega m} && \text{(since } s[m] = \hat{\phi}_{xx}[m]w[m]\text{)} \\
&= \sum_{m=0}^{N-1} \mathbb{E}(\hat{\phi}_{xx}[m])w[m]e^{-j\omega m} && \text{(window is deterministic)} \\
&= \sum_{m=0}^{N-1} \phi_{xx}[m]w[m]e^{-j\omega m} && \text{(since } \hat{\phi}_{xx}[m] \text{ is unbiased)} \\
&= \frac{1}{2\pi} P(\omega) * W(e^{j\omega}) && \text{(product in time} \implies \text{convolution in frequency)}
\end{aligned}
$$

Conclusions:

- *In average*, the Blackman-Tukey PSD estimate is equal to $P(\omega)$ *smeared* by the window $W(e^{j\omega})$.
- For the previous methods: $\mathbb{E}(P(\omega)) = \frac{1}{2\pi} P(\omega) * |W(e^{j\omega})|^2$
- If $W(e^{j\omega})$ takes on negative values, the PSD estimate could be negative. Thus, non-negative windows are preferred e.g, Bartlett

Blackman-Tukey PSD estimate in Matlab

```
>> phi_hat = xcorr(x, x, M-1, 'unbiased')
>> s = phi_hat.*w
>> s_e = ifftshift(s)
>> Pw = abs(fftshift(fft(s_e)))
```

(unbiased autocorrelation estimate)

(windowing)

(form DFT-even sequence)

(two-sided PSD in interval $[-\pi, \pi)$)

As you'll see in HW6, `xcorr` can be computed using FFT/IFFT with complexity $\mathcal{O}(N \log_2 N)$.

Example

See Matlab script: Canvas/Files/Matlab/`blackman_tukey_example.m`. This script requires the function Canvas/Files/Matlab/`blackman_tukey_psd.m`.

Conclusions:

- Increasing the number of samples Q improves accuracy
- Reducing the window length $L = 2M - 1$ improves accuracy at the expense of poorer resolution of the PSD
- The FFT length N determines the number of samples of the estimated PSD

Summary

- The PSD is the DTFT of the autocorrelation function
- The PSD may be two-sided or one-sided. Careful with conventions!
- The periodogram method estimates the PSD directly from the magnitude squared of the DFT of the windowed signal
- The periodogram is an biased estimator of the PSD, and it has large variance. Hence, the periodogram must be averaged to produce useful estimates
- The Welch method averages several periodograms
- The Welch method breaks the data into overalapping segments, each of length L. Usually, the segments overlap by $L/2$.
- Differently from the periodogram and Welch method, the Blackman-Tukey method estimates the PSD by computing the DFT of the estimated autocorrelation function
- Although the estimator of the autocorrelation function may be unbiased, the PSD estimate is biased. Windows with non-negative frequency response are typically preferred e.g., Bartlett
- Increasing the sequence length Q improves accuracy. Reducing the window length L improves accuracy at the expense of poorer frequency resolution.