Problem 1

(a) (8 points)

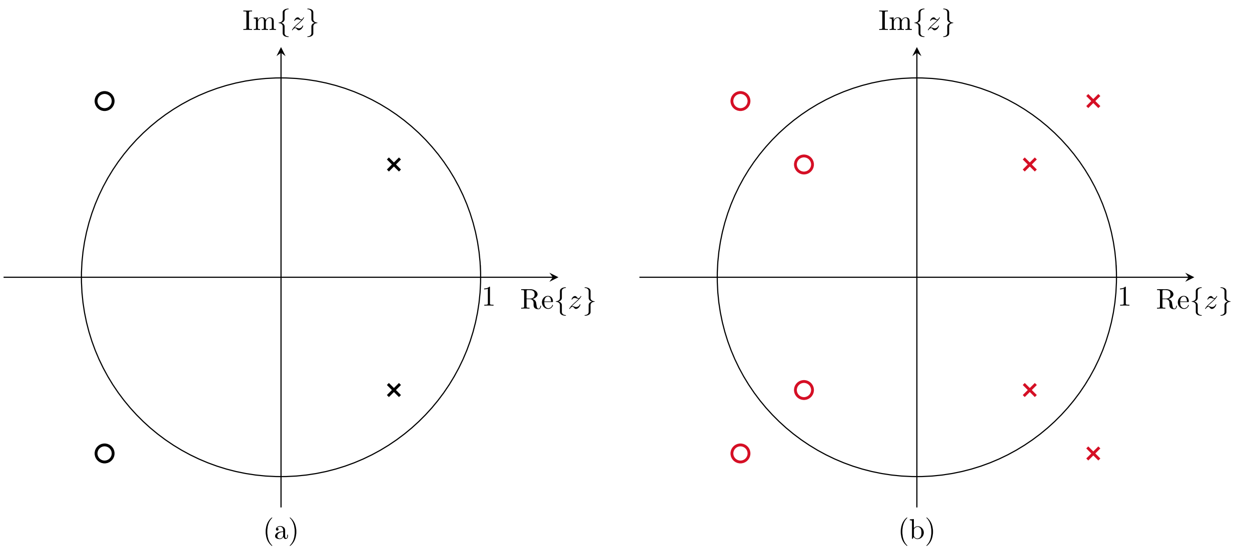


Figure 1: Pole-zero diagram of (a) $H(z)$ and (b) $C_{hh}(z)$.

Using the conjugate and time reversal properties of the z-transform, we can write: $h^*[-n] \iff H^*(1/z^*)$. Consequently,

$$C_{hh}(z) = H(z)H^*(1/z^*)$$

From this equation we see that all the poles and zeros of $H(z)$ are also poles and zeros of $C_{hh}(z)$. Moreover, suppose that p is a pole of $H(z)$. Then, $H(z)$ must have a factor $\frac{1}{z-p}$:

$$\frac{1}{z-p} \implies \frac{1}{(1/z^* - p)^*} = \frac{z}{p^*(1/p^* - z)} \tag{1}$$

Therefore, for each pole p of $H(z)$, $C_{hh}(z)$ has a pole at p, another pole at $1/p^*$, and a new zero at $z = 0$. We can use the same arguments to show that for every zero c of $H(z)$, $C_{hh}(z)$ has a zero at c, another zero at $1/c^*$, and a new pole at $z = 0$. Now we're ready to draw the zero-pole plot

(b) (4 points)

$c_{hh}[n]$ is not causal, since it is the convolution of a causal ($h[n]$) and an anti-causal ($h^*[-n]$) sequence. Therefore, the ROC of $C_{hh}(z)$ is the annulus (ring-shaped region) between two circles defines by the poles. This region includes the unit circle, and therefore $C_{hh}(z)$ is stable.

(c) (7 points)

Near $\pi/4$ (45 degrees) there is a pole in $H(z)$, therefore we should expect an increase in the magnitude response. There is a zero near $3\pi/4$ (135 degrees), therefore the magnitude response should decrease further near the zero.

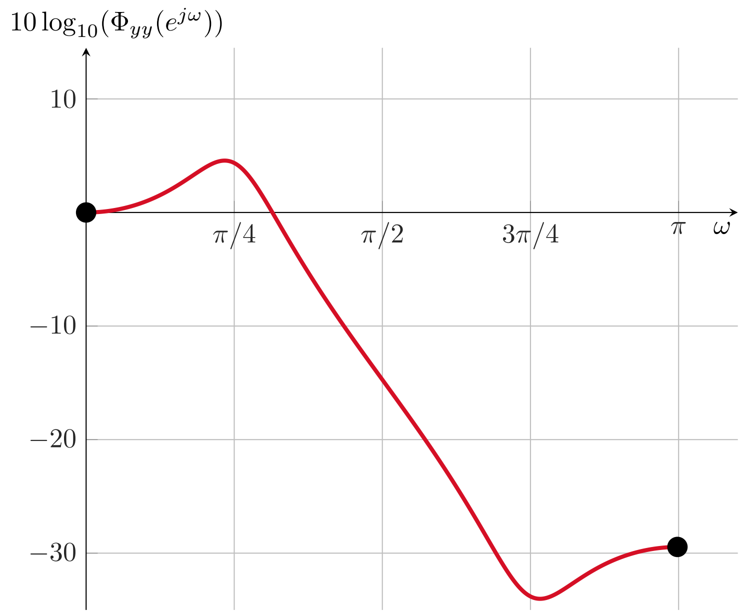


Figure 2: Sketch of the PSD of the output of $h[n]$ for a white noise input.

(d) (3 points)

$$\sigma_y^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{yy}(e^{j\omega})d\omega = \sigma_x^2\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2 d\omega$$
$$= \sigma_x^2 c_{hh}[0]$$

(e) (8 points)

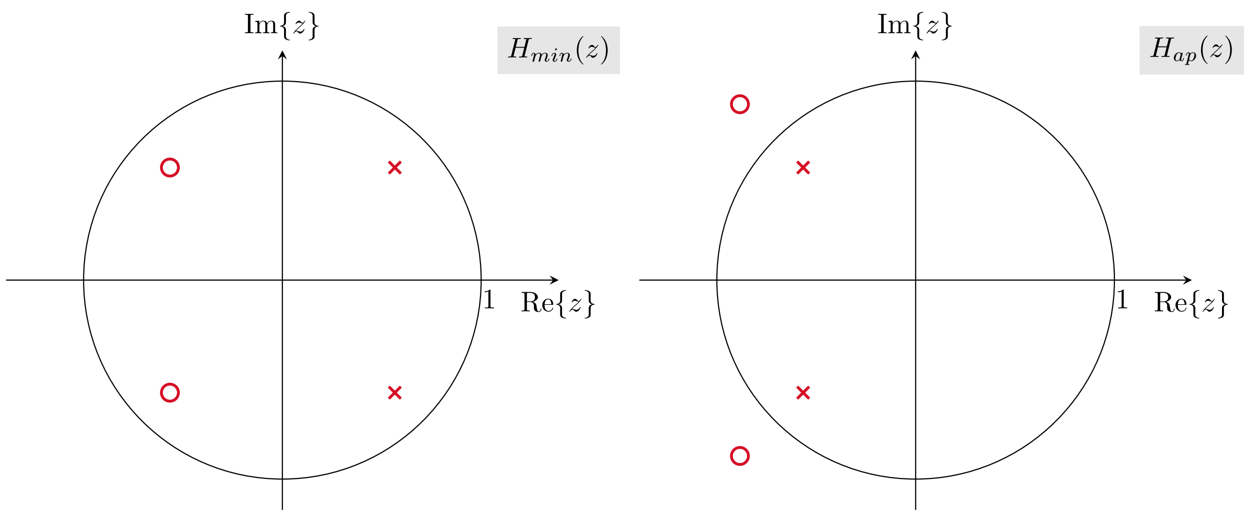
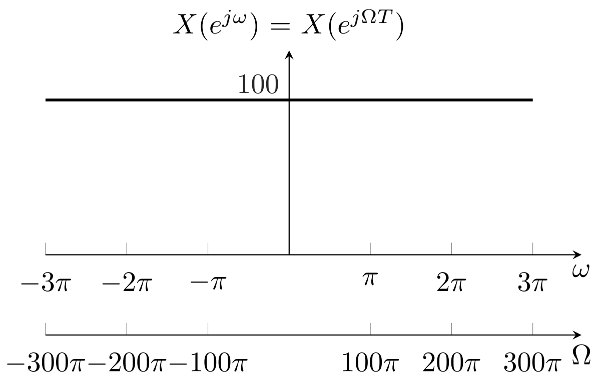


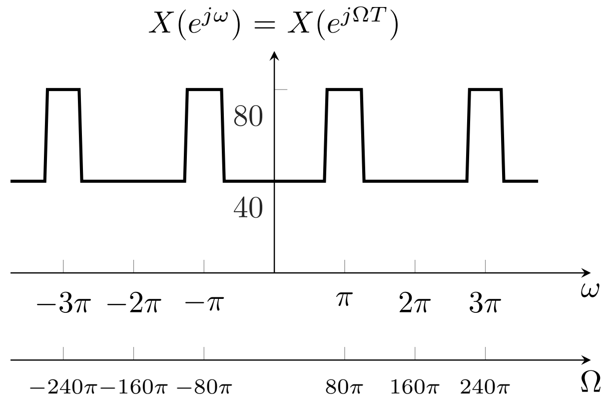
Figure 3: Minimum-phase and all-pass decomposition of $H(z)$.

Problem 2

(a)



(a) Scenario 1: $\Omega_s = 200\pi$

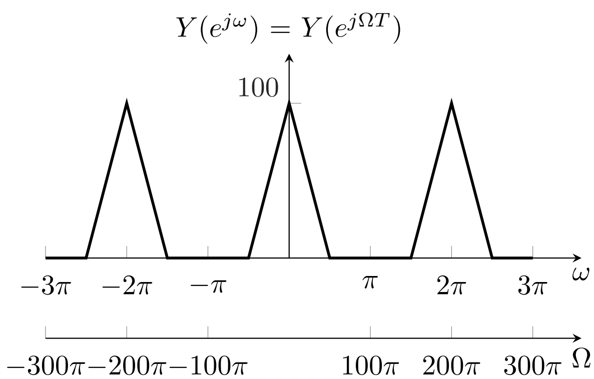


(b) Scenario 2: $\Omega_s = 160\pi$

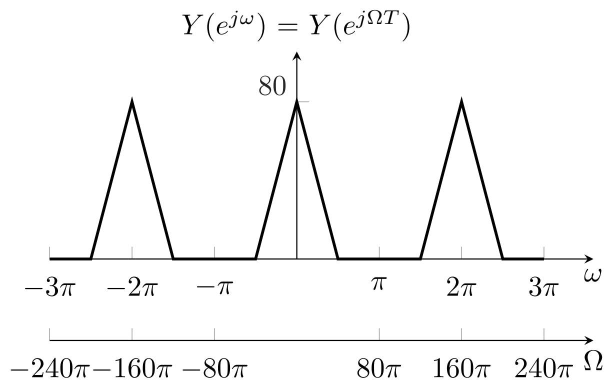
Figure 4: Fourier transform of the discrete-time signal $x[n]$ when (a) $\Omega_s = 200\pi$ and (b) $\Omega_s = 160\pi$.

(b)

To obtain $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$, we just need to multiply the result obtained in part (a) and $H(e^{j\omega})$:



(a) Scenario 1: $\Omega_s = 200\pi$



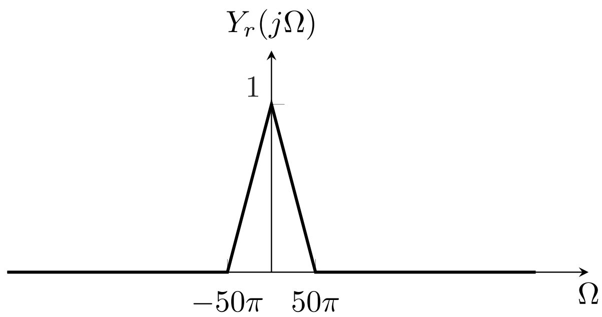
(b) Scenario 2: $\Omega_s = 160\pi$

Figure 5: Output spectrum obtained at the output of the discrete-time LTI system when (a) $\Omega_s = 200\pi$ and (b) $\Omega_s = 160\pi$.

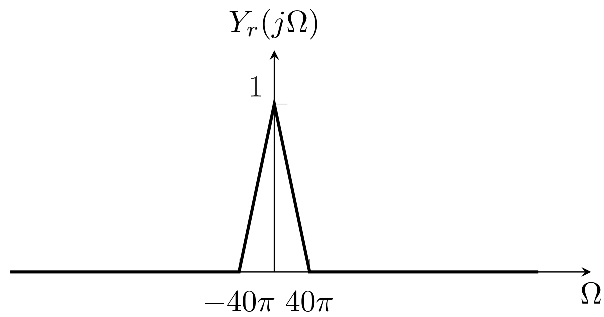
Note that, expect for a scaling factor, the two spectra are the same when compared with respect to the normalized frequency ω. That is, both of them have maximum frequency $\pi/2$. On the other hand, when compared with respect to the actual frequency Ω, the spectra differ. Specifically, in the first scenario the output spectrum has maximum frequency 50π, whereas in the second scenario the output spectrum has maximum frequency 40π.

(c)

After reconstruction with the ideal lowpass filter with cutoff frequency $\Omega_s/2$:



(a) Scenario 1: $\Omega_s = 200\pi$



(b) Scenario 2: $\Omega_s = 160\pi$

Figure 6: Output spectrum after reconstruction when (a) $\Omega_s = 200\pi$ and (b) $\Omega_s = 160\pi$.

Both spectra have amplitude 1, since, by definition, the ideal lowpass filter has gain T at frequency zero.

Note that even though the DSP operation was the same in both scenarios, the result in continuous time was different. This problem illustrates that the outcome depends on the sampling frequency.

Problem 3

(a)

To obtain a white noise spectrum after sampling, the sampling period T must be such that the linear taper part of the spectrum replicas perfectly overlaps as illustrated in the figure below:

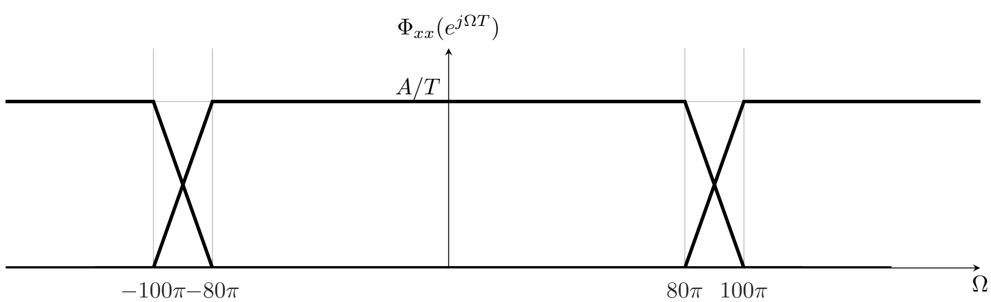


Figure 7: Output spectrum after reconstruction when (a) $\Omega_s = 200\pi$ and (b) $\Omega_s = 160\pi$.

This occurs when $\Omega_s = 180\pi = 2\pi/T$. Therefore, $T = 1/90$.

(b)

From Figure 7, we clearly see that

$$\sigma_x^2 = \frac{A}{T} = 90A \tag{2}$$

(c)

To obtain a white discrete-time PSD, we must have

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2, \text{ for any } \omega \tag{3}$$

We can write an equivalent condition for the autocorrelation function by calculating the inverse Fourier transform:

$$\phi_{xx}[m] = \sigma_x^2 \delta[m] = \begin{cases} \sigma_x^2, & m = 0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Since the discrete-time autocorrelation function is simply the sampled continuous-time autocorrelation function ($\phi_{xx}[m] = \phi_{x_c x_c}(mT)$), it follows that

$$\phi_{x_c x_c}(mT) = \sigma_x^2 \delta[m] = \begin{cases} \sigma_x^2, & m = 0 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Therefore, we can only obtain a discrete-time white noise from a non-white continuous-time noise if the continuous-time autocorrelation function is zero at the sampling instants $\tau = mT$, except at $\tau = mT = 0$.

Problem 4

(a)

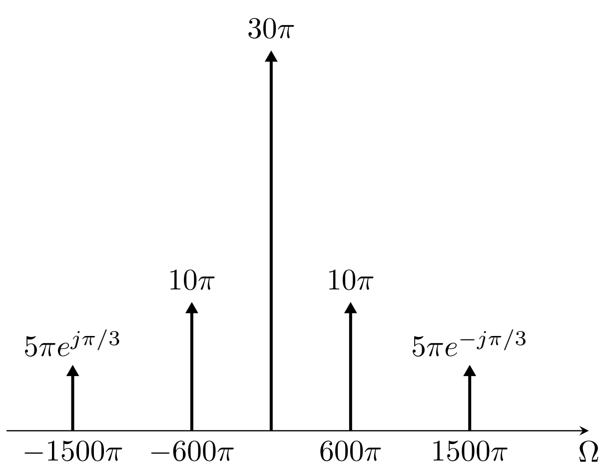
Recall the following Fourier transform pairs:

$$a(t) = c \Longleftrightarrow A(j\Omega) = 2\pi\delta(\Omega) \quad \text{(constant)}$$

$$a(t) = b\cos(\Omega_0 t - \phi) \Longleftrightarrow A(j\Omega) = b\pi(\delta(\Omega - \Omega_0)e^{-j\phi} + \delta(\Omega + \Omega_0)e^{j\phi}) \quad \text{(cosine)}$$

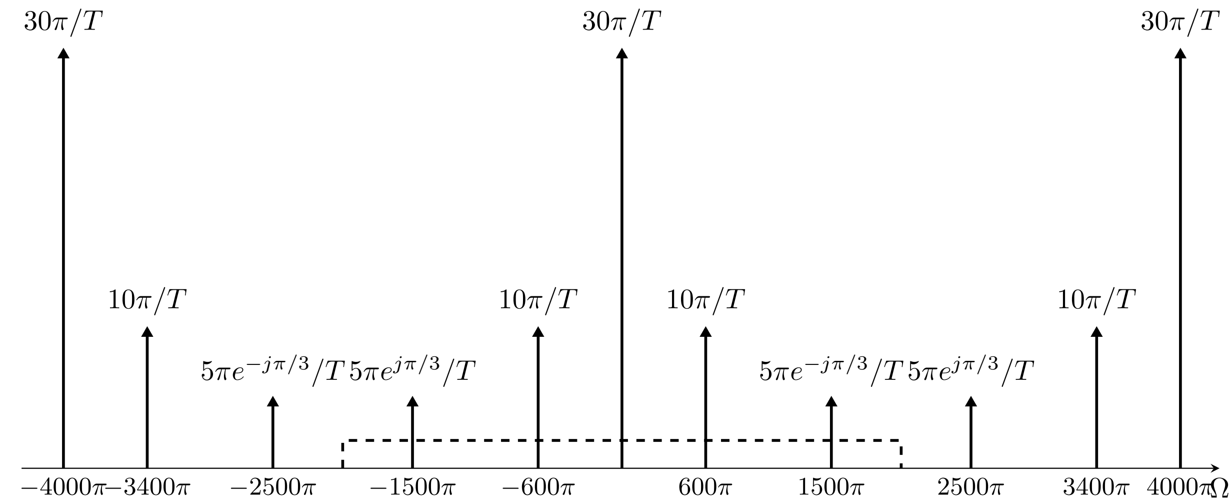
Now we can write an equation for $X_c(j\Omega)$,

$$X_c(j\Omega) = 30\pi\delta(\Omega) + 10\pi(\delta(\Omega - 600\pi) + \delta(\Omega + 600\pi)) + 5\pi(\delta(\Omega - 1500\pi)e^{-j\pi/3} + \delta(\Omega + 1500\pi)e^{j\pi/3}) \tag{6}$$



(b)

After sampling, there'll be replicas of the spectrum at multiples of $\Omega_s = 2\pi/T = 4000\pi$. Since the original signal is band-limited with highest frequency is $1500\pi < \Omega_s/2$, there will not be spectrum overlapping and aliasing distortion:



The reconstruction filter is indicated by the dashed line in the figure above.

The reconstructed signal is given by

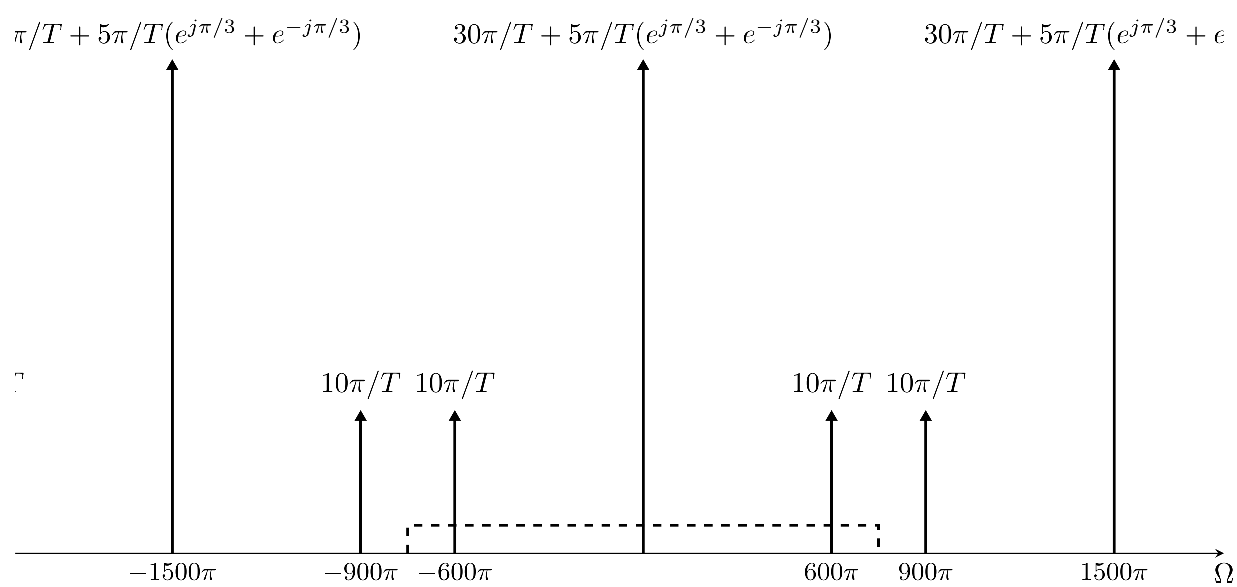
$$X_r(j\Omega) = 30\pi\delta(\Omega) + 10\pi(\delta(\Omega - 600\pi) + \delta(\Omega + 600\pi)) + 5\pi(\delta(\Omega - 1500\pi)e^{-j\pi/3} + \delta(\Omega + 1500\pi)e^{j\pi/3}) \quad (7)$$

and in the time domain:

$$X_r(t) = 15 + 10\cos(600\pi t) + 5\cos(1500\pi t - \pi/3) = x_c(t) \qquad \text{(exactly equal to the original signal)}$$

(c)

By choosing $T = 1/750$, we obtain the following spectrum. Note that the components of the term $5\cos(1500\pi t - \pi/3)$ now fall on the origin.



After the reconstruction filter depicted in the image above by the dashed lines, we obtain:

$$\begin{aligned} X_r(j\Omega) &= (30\pi + 5\pi(e^{j\pi/3} + e^{-j\pi/3}))\delta(\Omega) + 10\pi(\delta(\Omega - 600\pi) + \delta(\Omega + 600\pi)) \\ &= (30\pi + 10\pi\cos(\pi/3))\delta(\Omega) + 10\pi(\delta(\Omega - 600\pi) + \delta(\Omega + 600\pi)) \\ &= 35\pi\delta(\Omega) + 10\pi(\delta(\Omega - 600\pi) + \delta(\Omega + 600\pi)) \end{aligned} \tag{8}$$

and in time domain:

$$X_r(t) = 35/2 + 10\cos(600\pi t) \tag{9}$$

Problem 5

(a)

The maximum value of t_d will occur when the sound source is along the dashed line and above the first (top) microphone. This way, the sound will reach microphone 1, and only after propagating the distance d, it will reach the second microphone. Similarly, the minimum negative value of t_d will occur when the sound source is along the dashed line, but now below the second microphone. Therefore,

$$-\frac{d}{c} \leq t_d \leq \frac{d}{c} \tag{10}$$

(b)

$$\begin{aligned} \phi_{x_1x_2}[m] &= \mathbb{E}\{x_1[n+m]x_2[n]\} \\ &= \mathbb{E}\{(\alpha_1 s[n+m] + v_1[n+m])(\alpha_2 s_D[n] + v_2[n])\} \\ &= \mathbb{E}\{(\alpha_1 s[n+m] + v_1[n+m])(\alpha_2 s[n+D] + v_2[n])\} \\ &= \alpha_1\alpha_2\,\mathbb{E}(s[n+m]s[n+D]) + \alpha_1\,\mathbb{E}(s[n+m]v_2[n]) + \alpha_2\,\mathbb{E}(s[n+D]v_1[n+m]) + \mathbb{E}(v_2[n]v_1[n+m]) \end{aligned} \tag{11}$$

From the assumption that the signal and noises are all statistically independent, the last three terms are zero. Therefore,

$$\begin{aligned} \phi_{x_1x_2}[m] &= \alpha_1\alpha_2\,\mathbb{E}(s[n+m]s[n+D]) \\ &= \alpha_1\alpha_2\phi_{ss}[m-D] \end{aligned} \tag{12}$$

(c)

$$\begin{aligned} \Phi_{x_1x_2}(e^{j\omega}) &= \mathcal{F}\{\phi_{x_1x_2}[m]\} = \alpha_1\alpha_2\mathcal{F}\{\phi_{ss}[m-D]\} \\ &= \alpha_1\alpha_2\Phi_{ss}(e^{j\omega})e^{-j\omega D}, \end{aligned} \tag{13}$$

where the last equality follows from the time-delay property of the DTFT.

(d)

From part (b) $\phi_{x_1x_2}[m] = \alpha_1\alpha_2\phi_{ss}[m-D]$. Since $\phi_{ss}[n]$ is an autocorrelation function, its maximum value occurs when $n=0$. Therefore, the maximum value of $\phi_{x_1x_2}[m]$ will occur when $m=D$.

In an algorithm implementation, we can estimate $\phi_{x_1x_2}[m]$ from measurement, and find for which value of m, $\phi_{x_1x_2}[m]$ is maximized:

$$m^\star = \text{argmax}\phi_{x_1x_2}[m]. \tag{14}$$

Then, we can finally estimate $t_d = m^\star T$.

(e)

Using the result from part (c):

$$\Phi_{x_1x_2}(e^{j\omega}) = \alpha_1\alpha_2\Phi_{ss}(e^{j\omega})e^{-j\omega D} = \alpha_1\alpha_2\sigma_s^2 e^{-j\omega D} \tag{15}$$

We can calculate the cross-correlation function by simply taking the inverse DTFT of $\Phi_{x_1x_2}(e^{j\omega})$:

$$\begin{aligned}
\phi_{x_1x_2}[m] &= \int_{-\pi}^{\pi} \Phi_{x_1x_2}(e^{j\omega})e^{j\omega m}d\omega \\
&= \alpha_1\alpha_2\sigma_s^2 \int_{-\pi}^{\pi} e^{-j\omega D}e^{j\omega m}d\omega && (16)\\
&= \alpha_1\alpha_2\sigma_s^2 \int_{-\pi}^{\pi} e^{j\omega(m-D)}d\omega && (17)\\
&= \alpha_1\alpha_2\sigma_s^2 \left[\frac{1}{j(m-D)}e^{j\omega(m-D)}\right]_{-\pi}^{\pi} && (18)\\
&= \alpha_1\alpha_2\sigma_s^2 \frac{\sin(\pi(m-D))}{\pi(m-D)} && (19)\\
&= \alpha_1\alpha_2\sigma_s^2 \text{sinc}(m-D) && (20)
\end{aligned}$$

If D is an integer, then $\text{sinc}(m-D)$ is only non-zero when $m=D$. Therefore, we can write:

$$\phi_{x_1x_2}[m] = \begin{cases} \alpha_1\alpha_2\sigma_s^2\delta[m-D], & \text{when } D \text{ is integer} \\ \alpha_1\alpha_2\sigma_s^2\text{sinc}(m-D), & \text{otherwise} \end{cases}$$