Problem 1

(a)

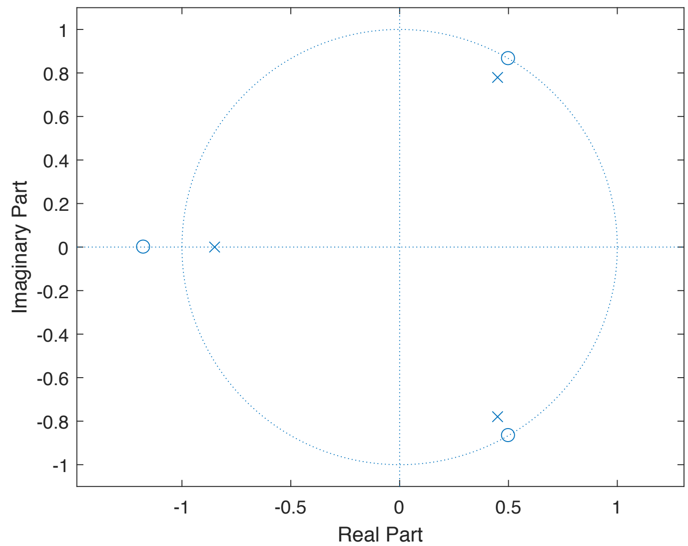


Figure 1: Zero-pole diagram

(b)

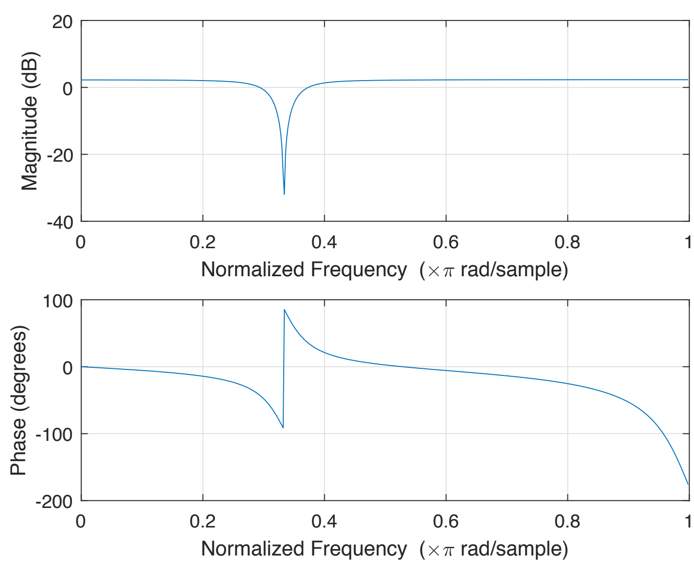


Figure 2: Magnitude and phase plots

(c)

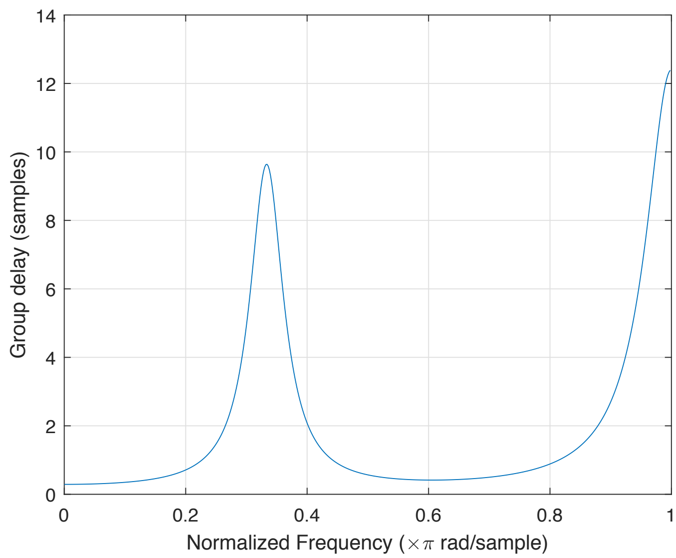


Figure 3: Group delay

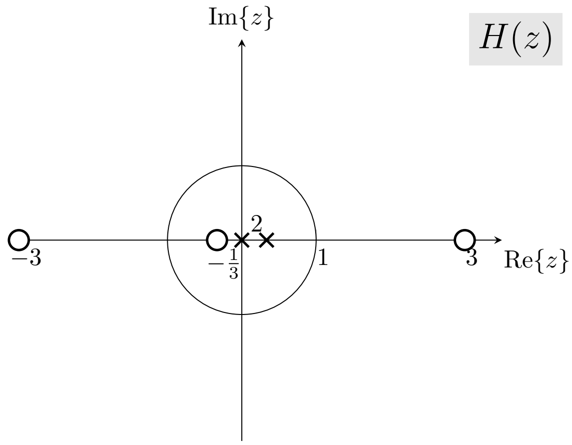
(d)

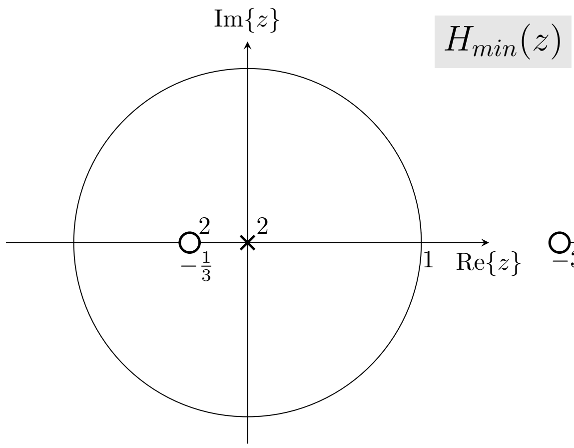
1. True. All poles are inside the unit circle.
2. False. Since the system is stable its impulse response must be absolute summable. Therefore, it cannot converge to a non-zero value when $n \to \infty$.
3. False. The system also has a zero at that frequency, and that zero is actually on the unit circle, so it has a bigger influence on the frequency response. As a result, the magnitude response has a dip at $\omega = \pi/3$.
4. False. The system has zeros outside the unit circle.
5. False. Since the system has a zero outside the unit circle, its inverse is unstable.

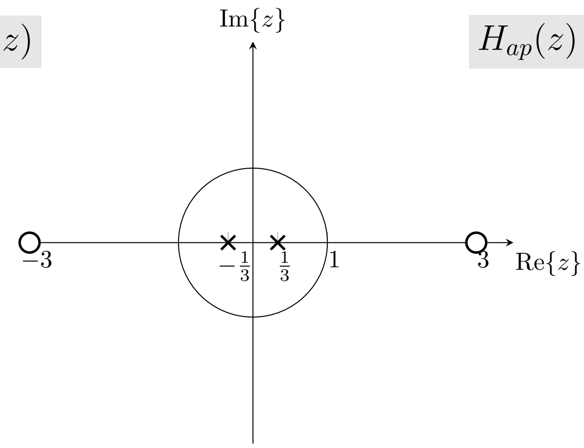
```
%% Problem 1 Code - a, b, c
b = conv(conv([1 -exp(1j*pi/3)], [1 -exp(-1j*pi/3)]), [1 1/0.85]);
a = conv(conv([1 -0.9*exp(1j*pi/3)], [1 -0.9*exp(-1j*pi/3)]), [1 0.85]);

%% a)
zplane(b, a) % b and a must be row vectors
saveas(gca, '../figs/hw4q1a_pole_zero', 'epsc')

%% b)
figure, freqz(b, a)
saveas(gca, '../figs/hw4q1b_mag_phase', 'epsc')

%% c)
figure, grpdelay(b, a)
saveas(gca, '../figs/hw4q1b_grpdelay', 'epsc')
```

Problem 2

(a)







We must follow this algorithm:

1. In the minimum phase system, reflect all zeros outside the unit circle to their reciprocal conjugate.
2. In the all-pass system, add poles at the reciprocal conjugate of the zeros outside the unit circle
3. (optional) Adjust gain of the minimum phase system to match the gain of the original system, so that the all-pass system has unit gain.

Therefore,

$$H_{min}(z) = -9(1 + 1/3z^{-1})^2 \tag{1}$$

$$H_{ap}(z) = -\frac{1}{9}\frac{(1 + 3z^{-1})(1 - 3z^{-1})}{(1 + 1/3z^{-1})(1 - 1/3z^{-1})} \tag{2}$$

This choice is unique up to a scale factor.

(b)

Yes, it has all poles at the origin.

(c)

Since the filter has real coefficients, if c is a zero, c^* must also be a zero.

Due to the symmetry conditions, causal generalized linear phase FIR systems must obey one of following relations:

$$H(z) = z^M H(z^{-1}) \qquad \text{(if even symmetric)}$$
$$H(z) = -z^M H(z^{-1}) \qquad \text{(if odd symmetric)}$$

Therefore, if c is a zero, $1/c$ must also be a zero:

$$H(c) = 0 = \pm c^M H(c^{-1})$$

Again, from the real coefficients condition, if $1/c$ is a zero, $1/c^*$ must also be a zero.

Therefore, if c is a zero, $c^*, 1/c, 1/c^*$ must also be zeros.

Note: this result can also be extended to other systems with real coefficients and even/odd symmetry. First, if a system (FIR or not) has real coefficients, its poles and zeros will appear as complex conjugate pairs. This follows from algebra, since the roots of a polynomial with real coefficients are either real or appear in complex conjugate pairs. Second, if a system (FIR or not) has even or odd symmetry, the poles and zeros will appear as conjugate reciprocal pairs. This follows from the properties of the z-transform:

$$h[n] = \pm h[-n] \qquad \text{(even or odd symmetry)}$$
$$H(z) = \pm H(z^{-1}) \qquad \text{(time reversal property of the } z\text{-transform)}$$

Therefore, if c is a zero of $H(z)$, $1/c$ must also be a zero of $H(z)$, since $H(z = 1/c) = H(z = c) = 0$. Analogously, if p is a pole of $H(z)$, $1/p$ must also be a pole of $H(z)$, since $H(z = 1/p) = H(z = p) = \infty$.

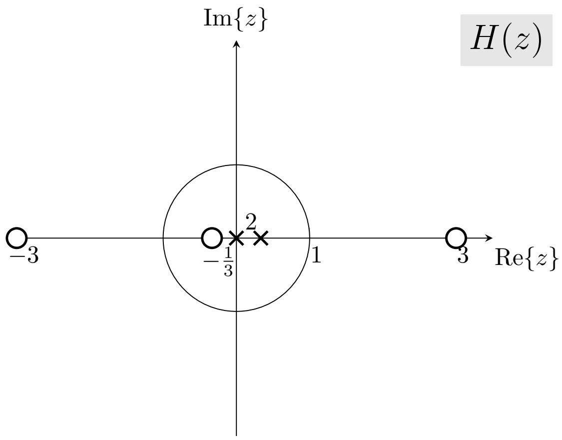
(d)

Using the result from part (c), we can do the following decomposition:

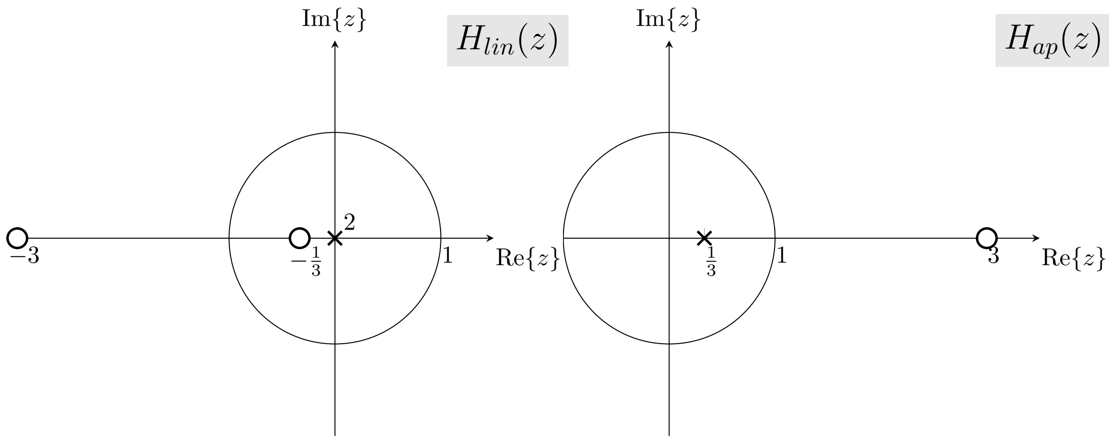
Therefore,

$$H_{lin}(z) = (1 + 3z^{-1})(1 + 1/3z^{-1}) \tag{3}$$

$$H_{ap}(z) = -\frac{1 - 3z^{-1}}{1 - 1/3z^{-1}} \tag{4}$$



Im{z}
H(z)
−3
2
−1/3
1
3
Re{z}



Im{z}
H_lin(z)
−3
2
−1/3
1
Re{z}
Im{z}
H_ap(z)
1/3
1
3
Re{z}

Problem 3

(a) A, B, and E. All their poles are at the origin.

(b) All. All of them have all poles inside the unit circle.

(c) F. It's the only system that appears to have zeros located at the conjugate symmetric of its poles i.e., if e_k is a pole, then $1/e_k^*$ must be a zero.

(d) E. All its zeros are inside the unit circle. [D is also acceptable as no zeroes are outside the unit circle.]

(e) A and B. Their zeros appear in conjugate pairs and conjugate reciprocal pairs.

(f) A and D. They have zeros at $z = -1$, which corresponds to $\omega = \pm\pi$.

(g) A and D. Their maximum frequency response appears at low frequencies. In A for the absence of zeros close to $\omega = 0$, and in D the poles near $\omega = 0$ increase the magnitude response.

(h) E. It's the FIR system with the fewest number of zeros (poles).

(i) E. It's the only minimum phase system.

(j)

$$\begin{aligned} H(z) &= \frac{(1+1.1z^{-1})(1-1.1z^{-1})}{(1+0.8z^{-1})(1-0.8z^{-1})} \\ &= \underbrace{\frac{(1+1/1.1z^{-1})(1-1/1.1z^{-1})}{(1+0.8z^{-1})(1-0.8z^{-1})}}_{H_{min}(z)} \cdot \underbrace{\frac{(1+1.1z^{-1})(1-1.1z^{-1})}{(1+1/1.1z^{-1})(1-1.1z^{-1})}}_{H_{ap}(z)} \end{aligned} \tag{5}$$

Problem 4: Kramers-Kronig relations in discrete time

(a)

Using the time reversal property of real signals

$$x[-n] \Longleftrightarrow X^*(e^{j\omega}) \qquad \text{(time reversal of real signals)}$$

We can write:

$$X_e(e^{j\omega}) = \frac{1}{2}\Big(X(e^{j\omega}) + X^*(e^{j\omega})\Big) = \text{Re}\{X(e^{j\omega})\} \tag{6}$$

$$X_o(e^{j\omega}) = \frac{1}{2}\Big(X(e^{j\omega}) - X^*(e^{j\omega})\Big) = j\text{Im}\{X(e^{j\omega})\} \tag{7}$$

These results indicate that for real-valued signals, even symmetry in time domain corresponds to purely real frequency component, whereas odd symmetry in time domain corresponds to purely imaginary frequency component. This relationship stems from the fact that $e^{j\omega} = \cos\omega + j\sin\omega$, cosine is a even function and sine is an odd function.

This result is also valid for continuous-time signals.

(b)

Since $x[n] = x_e[n] + x_o[n]$, and since $x[n]$ is causal i.e., $x[n] = 0, n < 0$, we must have that

$$x_o[n] = -x_e[n], n < 0 \implies x[n] = 0, n < 0 \tag{8}$$

By definition $x_o[n]$ is odd symmetric: $x_o[n] = -x_o[-n]$. Therefore,

$$\begin{aligned} x_o[n] &= \begin{cases} x_e[n], & n > 0 \\ 0, & n = 0 \\ -x_e[n], & n < 0 \end{cases} \\ &= \text{sign}[n] x_e[n], \end{aligned} \tag{9}$$

where the sign function is simply

$$\text{sign}[n] = \begin{cases} 1, & n > 0 \\ 0, & n = 0 \\ -1, & n < 0 \end{cases} \tag{10}$$

Multiplication in time domain corresponds to convolution in frequency domain:

$$X_o(e^{j\omega}) = \frac{1}{2\pi}\Big(\mathcal{F}\{\text{sign}[n]\} * X_e(e^{j\omega})\Big), \tag{11}$$

Following the hint: $\mathcal{F}\{\text{sign}[n]\} = -j\frac{\sin\omega}{1-\cos\omega}, |\omega| < \pi$. Thus,

$$X_o(e^{j\omega}) = \frac{1}{2\pi}\Big(-j\frac{\sin\omega}{1-\cos\omega} * X_e(e^{j\omega})\Big), \tag{12}$$

(c)

Combining the results from parts (a) and (b):

$$\begin{aligned} X_o(e^{j\omega}) &= \frac{1}{2\pi}\Big(-j\frac{\sin\omega}{1-\cos\omega} * X_e(e^{j\omega})\Big) && \text{(from part (b))} \\ j\text{Im}\{X(e^{j\omega})\} &= \frac{1}{2\pi}\Big(-j\frac{\sin\omega}{1-\cos\omega} * \text{Re}\{X(e^{j\omega})\}\Big) && \text{(from part (a))} \\ \text{Im}\{X(e^{j\omega})\} &= -\frac{1}{2\pi}\Big(\frac{\sin\omega}{1-\cos\omega} * \text{Re}\{X(e^{j\omega})\}\Big) && (13) \end{aligned}$$