Problem 1

(i) $y[n] = x[n] - 0.5x[n-1] + 0.5x[n-2]$

(a)

By applying the linearity and time shift properties of the z-transform:

$$Y(z) = X(z)(1 - 0.5z^{-1} + 0.5z^{-2})$$
$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} + 0.5z^{-2} \quad (1)$$
$$H(z) = \frac{z^2 - 0.5z + 0.5}{z^2} \quad (2)$$

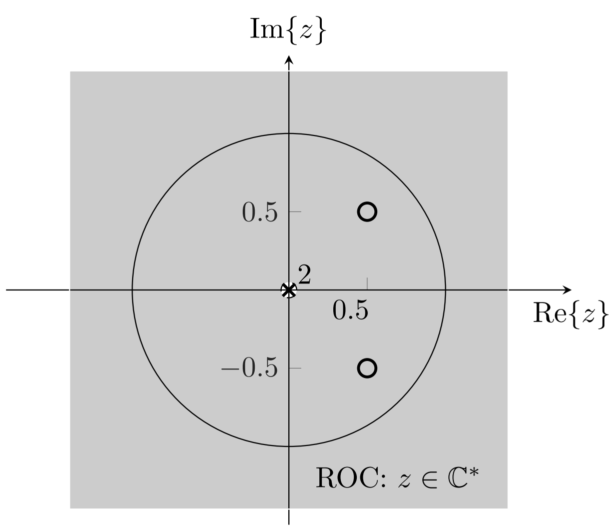
(b)

From the derivation above we can see that $H(z)$ has two poles at the origin ($p_1 = p_2 = 0$). We can use the function `roots([1, -1, 0.5])` to determine that $H(z)$ has two zeros $z_1 = 0.5 + j0.5$ and $z_1^* = 0.5 - j0.5$. Note that since the polynomial coefficients are real, complex roots appear in complex conjugate pairs.

We can determine that this system is causal by inspecting the difference equation and noticing that at any given time n, the output only depends on the current and past samples $\{x[n], x[n-1], x[n-2]\}$.

As the system is causal, the ROC is the exterior of a circle whose radius is the magnitude of the outermost pole. In this case, we only have poles at the origin. Hence, the ROC is the entire z-plan with the exception of $z = 0$. Hence, ROC $= \mathbb{C}^*$.

Since the ROC contains the unit circle, this system is stable.



(c)

```
>> freqz([1, -0.5, 0.5], 1)
```

Note that the Matlab uses the coefficients of polynomials of z^{-1}.

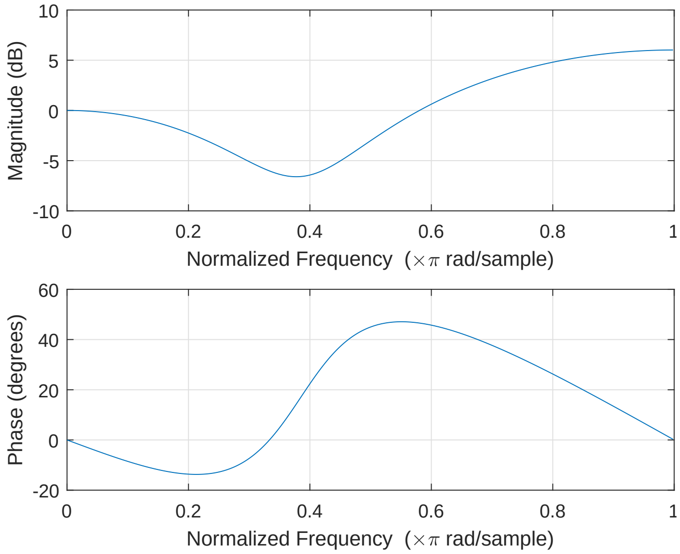


Figure 1: Maginute and phase response for the system defined by the difference equation (i).

(ii) $y[n]-0.5y[n-1]+0.2y[n-2]+0.3y[n-3] = x[n]+0.5x[n-1]+0.1x[n-2]+0.5x[n-3]$

(a)

By applying the linearity and time shift properties of the z-transform:

$$Y(z)(1-0.5z^{-1}+0.2z^{-2}+0.3z^{-3}) = X(z)(1+0.5z^{-1}+0.1z^{-2}+0.5z^{-3})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.5z^{-1}+0.1z^{-2}+0.5z^{-3}}{1-0.5z^{-1}+0.2z^{-2}+0.3z^{-3}} \tag{3}$$

$$H(z) = \frac{z^3+0.5z^2+0.1z+0.5}{z^3-0.5z^2+0.2z+0.3} \tag{4}$$

(b)

For the zeros:

```
roots([1, 0.5, 0.1, 0.5])
```

This results in $z_1 = 0.9494$, $z_2 = 0.2247 + j0.69$, $z_3 = z_2^* = 0.2247 - j0.69$.

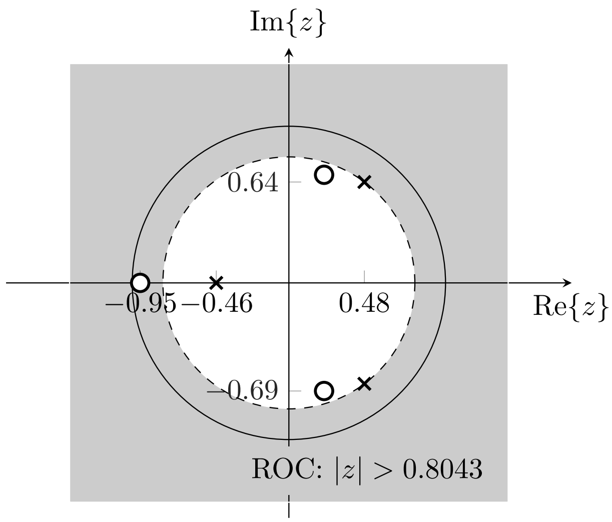
For the poles:

```
roots([1, -0.5, 0.2, 0.3])
```

This results in $z_1 = 0.9494$, $z_2 = 0.2247 + j0.69$, $z_3 = z_2^* = 0.2247 - j0.69$.

This system is also causal, since, from the difference equation, at any given time n, the output only depends on the current and past samples. In this case, the outermost poles are the complex conjugate pair with $|p_2| = 0.8043$. Therefore, ROC $= \{|z| > 0.8043\}$.

The ROC contains the unit circle, therefore this system is stable.



(c)

```
>> freqz([1, 0.5, 0.1, 0.5], [1, -0.5, 0.2, 0.3])
```

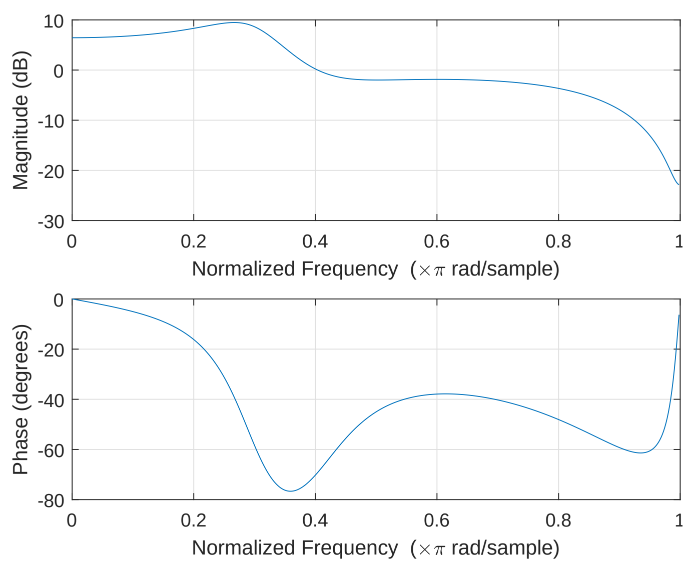


Figure 2: Maginute and phase response for the system defined by the difference equation (ii).

(d)

By inspecting the magnitude plots of both systems, we see that while system (i) has a gain of -6.4 dB at $\omega = 0.4\pi$, system (ii) has a gain of 0 dB at the same frequency. Therefore, system (ii) will produce the output with highest amplitude when the input is $x[n] = \cos(0.4\pi n)$.

Problem 2: Echo cancellation

(a)

For the first echo generating system:

$$Y(z) = X(z) + \alpha X(z) z^{-N} H_1(z) = \frac{Y(z)}{X(z)} = 1 + \alpha z^{-N}$$

For the second echo generating system:

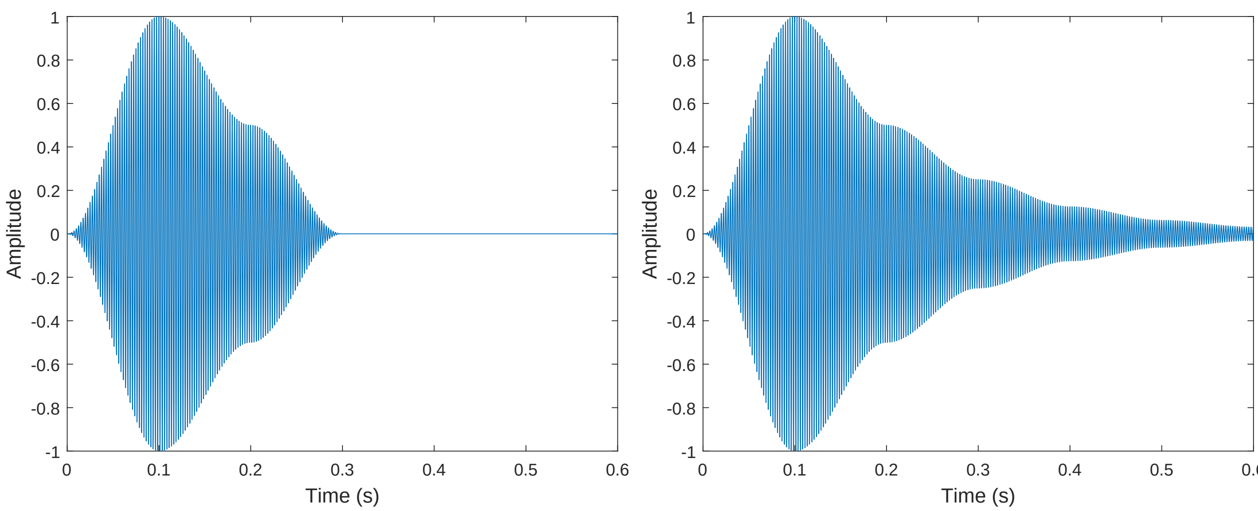
$$Y(z) = X(z) + \alpha Y(z) z^{-N} H_2(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \alpha z^{-N}}$$

From the problem statement, the first echo is delayed by 0.1 s. Since the sampling frequency is 8 kHz, this corresponds to

$$N = T_{echo} f_s = 0.1 \cdot 8 \times 10^3 = 800 \tag{5}$$

(b)

The code to generate these plots is included at the end of the solutions to this question.



(a) Echo-generating system 1

(b) Echo-generating system 2

Figure 3: Waveforms of the echoed signals generated by the echo-generating systems (a) 1 and (b) 2.

(c)

For each system, we just need to calculate its inverse:

$$G_1(z) = H_1^{-1}(z) = \frac{1}{1 + \alpha z^{-N}} \tag{system 1}$$

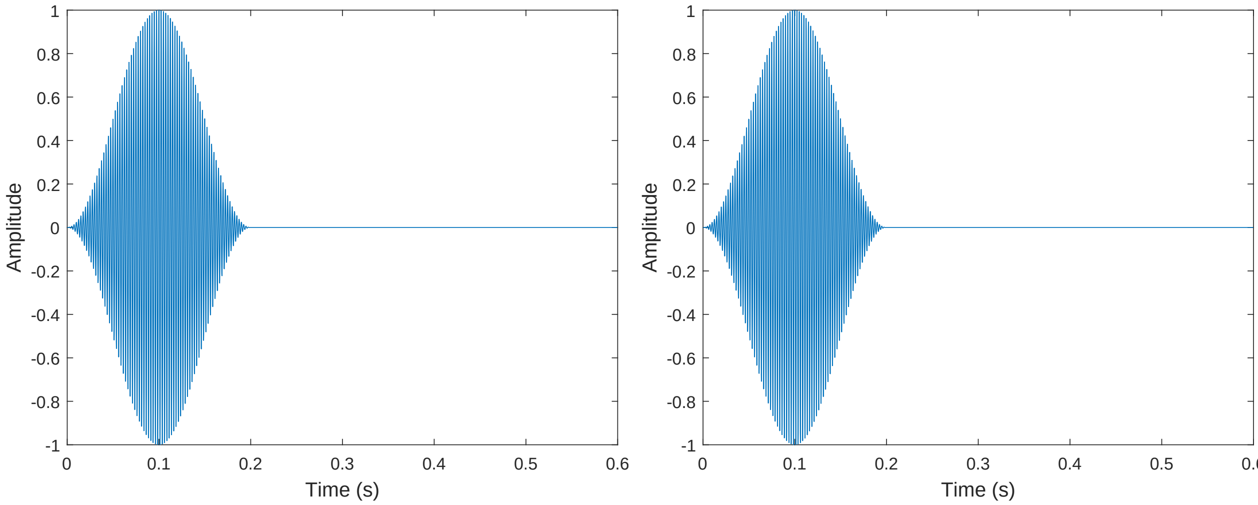
$$G_2(z) = H_2^{-1}(z) = 1 - \alpha z^{-N} \tag{system 2}$$

System 1 is IIR, as it has poles different from zero. System 2 is FIR, as all its poles are at zero.

Note that while system 2 is always stable, system 1 is only stable if $|\alpha| < 1$.

(d)

We clearly see from the images below that perfect echo cancellation was achieved for both systems. System 1 could fail only if $|\alpha| \geq 1$, in which case, the system could become unstable.



(a) Echo cancellation 1 (b) Echo cancellation 2

Figure 4: Recovered waveforms after echoed cancellation of echoed signals produced by the echo-generating systems (a) 1 and (b) 2.

Code for problem 2

```
%% Template for the echo cancelling problem
clear, clc, close all

fs = 8e3;                           % sampling frequency (Hz)
f = 400;                            % sinusoid frequency (Hz)
Tdur = 0.2;                         % pulse duration (s)
x = pulse(f,Tdur,fs);               % Generates a pulse of frequency f, duration Tdur, and s
x = [x zeros(1, 2*length(x))];      % zero pad for processing
n = 0:length(x)-1;                  % discrete-time vector
t = n/fs;                           % continuous-time vector

plot(t, x);                 % Plot signal
xlabel('Time (s)')
ylabel('Amplitude')
title('Original pulse: 400-Hz sinusoid, 200-ms Hann window')
sound(x, fs);               % Play the sound

%% Your code goes here
alpha = 0.5;
Techo = 0.1;
N = Techo*fs;
```

```
% Echo generation system 1
a1 = 1;
b1 = zeros(1, N+1); % b1 has N+1 coefficients
b1(1) = 1;
b1(N+1) = alpha;

y1 = filter(b1, a1, x);
figure
plot(t, y1);                  % Plot signal
xlabel('Time (s)', 'FontSize', 12)
ylabel('Amplitude', 'FontSize', 12)
saveas(gca, '../figs/hw01q2_echoed1', 'epsc')
sound(y1, fs);                % Play the sound

% Echo generation system 2
a2 = zeros(1, N+1);          % a1 has N+1 coefficients
a2(1) = 1;
a2(N+1) = -alpha;
b2 = 1;

y2 = filter(b2, a2, x);
figure
plot(t, y2);                  % Plot signal
xlabel('Time (s)', 'FontSize', 12)
ylabel('Amplitude', 'FontSize', 12)
saveas(gca, '../figs/hw01q2_echoed2', 'epsc')
sound(y2, fs);                % Play the sound

%% Echo cancellation
% System 1
xrec1 = filter(a1, b1, y1); % We just need to switch the coefficients (a1, b1)
figure
plot(t, xrec1);                  % Plot signal
xlabel('Time (s)', 'FontSize', 12)
ylabel('Amplitude', 'FontSize', 12)
saveas(gca, '../figs/hw01q2_echoed_rec1', 'epsc')
sound(xrec1, fs);                % Play the sound

% System 2
xrec2 = filter(a2, b2, y2);
figure
plot(t, xrec2);                  % Plot signal
xlabel('Time (s)', 'FontSize', 12)
ylabel('Amplitude', 'FontSize', 12)
```

```
saveas(gca, '../figs/hw01q2_echoed_rec2', 'epsc')
sound(xrec1, fs);                % Play the sound
```

Problem 3

(a)

Writing the transfer function of each system in a convenient form:

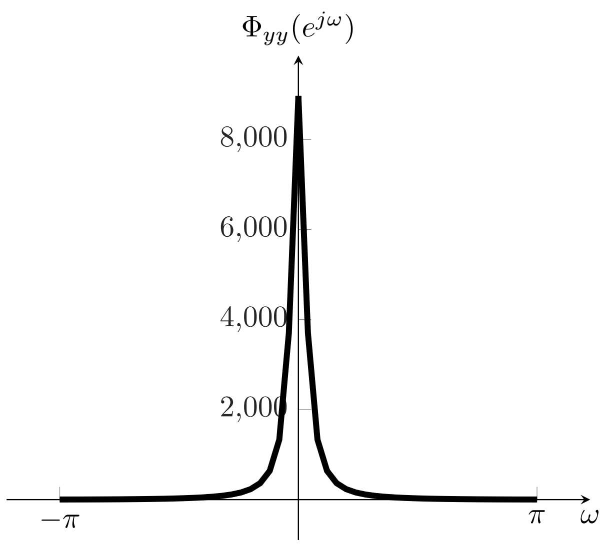
$$H_1(e^{j\omega}) = 1 + e^{-j\omega} \implies |H_1(e^{j\omega})|^2 = 2(1+\cos(\omega)) \tag{6}$$

$$H_2(e^{j\omega}) = \frac{1}{1-0.9e^{-j\omega}} \implies |H_2(e^{j\omega})|^2 = \frac{1}{1.81 - 1.8\cos(\omega)} \tag{7}$$

$$\tag{8}$$

Note that while signal $x[n]$ undergoes system $h_1[n] * h_2[n]$ (the cascade of system 1 and system 2), signal $e[n]$ undergoes only system $h_2[n]$. Therefore, by superposition,

$$\begin{aligned}\Phi_{yy}(e^{j\omega}) &= \Phi_{xx}(e^{j\omega})|H_1(e^{j\omega})H_2(e^{j\omega})|^2 + \Phi_{ee}(e^{j\omega})|H_2(e^{j\omega})|^2 \\ &= 40\frac{1+\cos(\omega)}{1.81-1.8\cos(\omega)} + \frac{10}{1.81-1.8\cos(\omega)} \\ &= \frac{50+40\cos(\omega)}{1.81-1.8\cos(\omega)}\end{aligned} \tag{9}$$



(b)

By definition,

$$\mathbb{E}(y^2[n]) = \phi_{yy}[0] = 452.6316 \tag{10}$$

Problem 4

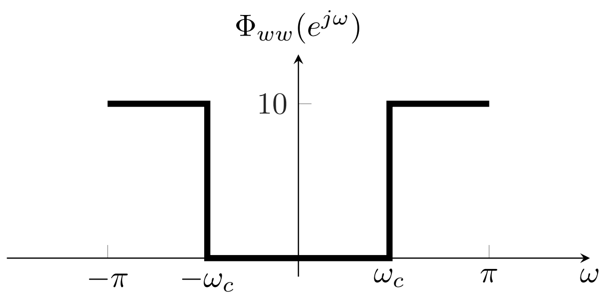
Note: The input $x[n]$ is a random signal with zero mean and autocorrelation $\phi_{xx}[m] = 20\delta[m]$ in the assignment, but the solution solves for autocorrelation $\phi_{xx}[m] = 10\delta[m]$.

(a)

From the diagram $w[n] = x[n] * h_1[n]$. Therefore,

$$\phi_{ww}[m] = \phi_{xx}[m] * c_{h_1 h_1}[m] = \phi_{xx}[m] * h_1[m] * h_1^*[-m] \tag{11}$$

$$\begin{aligned}\Phi_{ww}(e^{j\omega}) &= \Phi_{xx}(e^{j\omega}) \cdot |H_1(e^{j\omega})|^2 \\ &= 10 \cdot |H_1(e^{j\omega})|^2 &= \begin{cases} 0, & |\omega| < \omega_c \\ 10, & \omega_c < |\omega| < \pi \end{cases}\end{aligned} \tag{12}$$



(b)

By inspection, we can write the PSD of w as a constant minus the ideal lowpass filter from $-\omega_c$ to ω_c. The inverse DTFT of a constant is an impulse at the origin, and the inverse DTFT of the ideal lowpass filter is a sinc function.

$$\phi_{ww}[m] = 10\delta[m] - 10\frac{\sin \omega_c m}{\pi m} \tag{13}$$

(c)

$$\mathbb{E}(w^2[n]) = \phi_{ww}[0] = 10 - 10\frac{\omega_c}{\pi} = 10(1 - \frac{\omega_c}{\pi}) \tag{14}$$

(d)

For system 2 we have

$$h_2[m] = \delta[m-5] \longleftrightarrow H_2(e^{j\omega}) = e^{-j5\omega}. \tag{15}$$

Therefore,

$$\Phi_{yy}(e^{j\omega}) = \Phi_{ww}(e^{j\omega})|H_2(e^{j\omega})|^2 = \Phi_{ww}(e^{j\omega}) \tag{16}$$

Consequently, $\phi_{yy}[m] = \phi_{ww}[m]$.

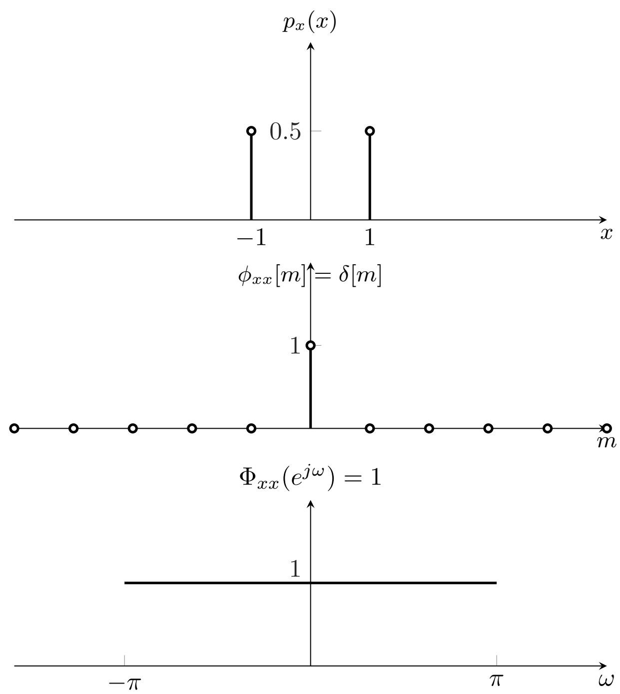
$$\mathbb{E}(y^2[n]) = \phi_{yy}[0] = \phi_{ww}[0] = 10(1 - \frac{\omega_c}{\pi}) \tag{17}$$

(e)

$$\begin{aligned}\phi_{wy}[m] &= \mathbb{E}(w[n]y[n+m]) \\ &= \mathbb{E}(w[n]w[n+m-5]) \qquad (18) \\ &= \phi_{ww}[m-5] \\ &= \begin{cases} 10(1-\frac{\omega_c}{\pi}), & m=5 \\ -10\frac{\sin\omega_c(m-5)}{\pi(m-5)}, & m \neq 5 \end{cases} \qquad (19)\end{aligned}$$

Problem 5

(a)

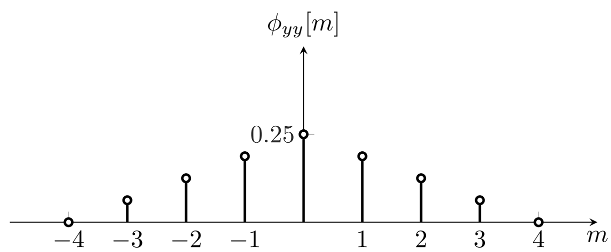


(b)

$$\begin{aligned}
\phi_{yy}[m] &= \mathbb{E}(y[m]y[n+m]) \\
&= \frac{1}{L^2}\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\mathbb{E}(x[n-k]y[n+m-l]) \\
&= \frac{1}{L^2}\sum_{k=0}^{L-1}\sum_{l=0}^{L-1}\phi_{xx}[m+k-l] \\
&= \frac{1}{L^2}\sum_{\substack{0\le k\le L-1\\0\le l\le L-1\\l-k=m}} 1 = \begin{cases}\frac{L-|m|}{L^2}, & |m|<L\\ 0, & \text{otherwise}\end{cases}
\end{aligned} \tag{20}$$

Note: The question in the assignment asks for a plot for L=5, but the concept remains the same and the plot for L=4 is shown below.

For $L = 4$,



(c)

For the mean of $y[m]$:

$$\begin{aligned}
\mu_y = \mathbb{E}(y[m]) &= \mathbb{E}\left(\frac{1}{L}\sum_{k=0}^{L-1}x[m-k]\right) \\
&= \frac{1}{L}\sum_{k=0}^{L-1}\mathbb{E}(x[m-k]) &= 0
\end{aligned}$$

For the average power of $y[m]$:

$$\mathbb{E}(y^2[m]) = \phi_{yy}[0] = \frac{1}{L}$$