Stanford University
EE 264: Digital Signal Processing
Summer, 2018

Homework #03

Date assigned: July 13, 2018 Date due: July 20, 2018

Reading: Problems 2-5 need lecture notes 4, which will be finished in class on Monday, July 16, 2018.
Homework submission: Please submit your solutions on Gradescope. Create a single .pdf file containing all your analytical derivations, sketches, plots, and Matlab code (if any).

Problem 1: Poles and zeros of the deterministic autocorrelation function (30 points)

As discussed in class, the deterministic autocorrelation function of an LTI system with impulse response $h[n]$ is defined by

$$c_{hh}[n] = h[n] * h^*[-n]. \tag{1}$$

For this problem, consider that $h[n]$ is <u>causal</u> and that the pole-zero diagram of its z-transform, $H(z)$, is shown in Figure 1a.

(a) (8 points) Sketch the pole-zero diagram of the z-transform of the deterministic autocorrelation function $C_{hh}(z) \Longleftrightarrow c_{hh}[n]$. Use the empty diagram of Figure 1b for your sketch.

Hint: First write $C_{hh}(z)$ in terms of $H(z)$ using the properties of the z-transform.

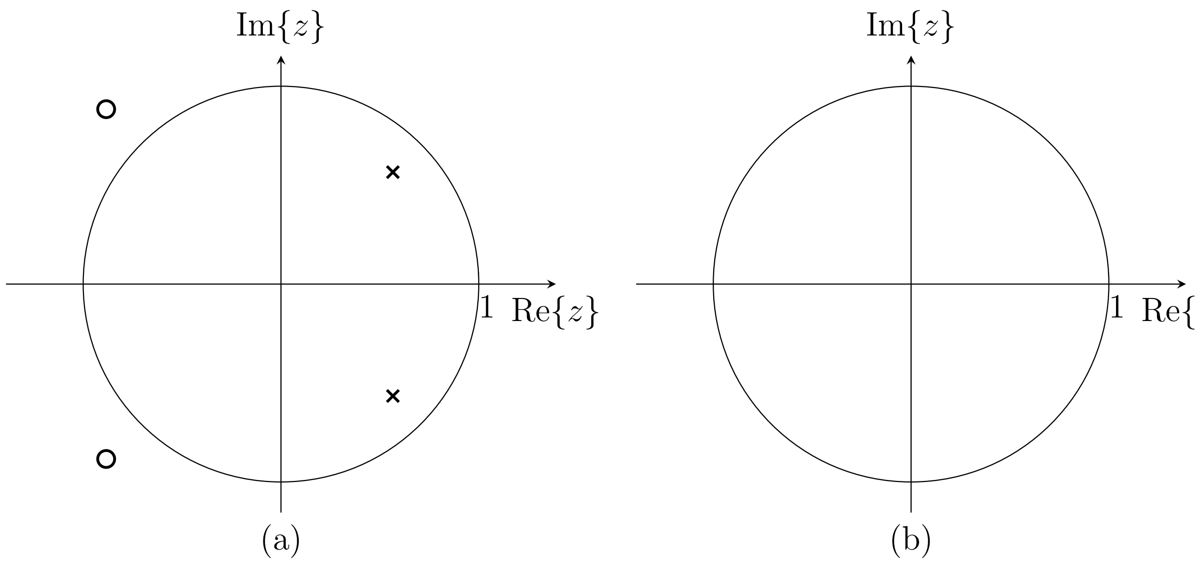


Figure 1: Pole-zero diagram of (a) $H(z)$ and (b) $C_{hh}(z)$.

(b) (4 points) In part (a) you may have noticed that $C_{hh}(z)$ has poles outside the unit circle. Does that mean that $C_{hh}(z)$ is unstable? Justify your answer.

Hint: $h[n]$ is causal, but is $c_{hh}[n]$ causal?

(c) (7 points) Suppose that a <u>white</u> noise signal $x[n]$ with average power $\sigma_x^2 = 1$ is input to the system $h[n]$. Sketch the PSD of the output noise $y[n] = x[n] * h[n]$. Since the pole-zero diagram does not provide information about the gain, the first and last points of the magnitude response are already included. Note that the magnitude in this plot is in log scale.

Note: your sketch does not need to be perfect to receive full credit, but be sure to indicate the most relevant points.

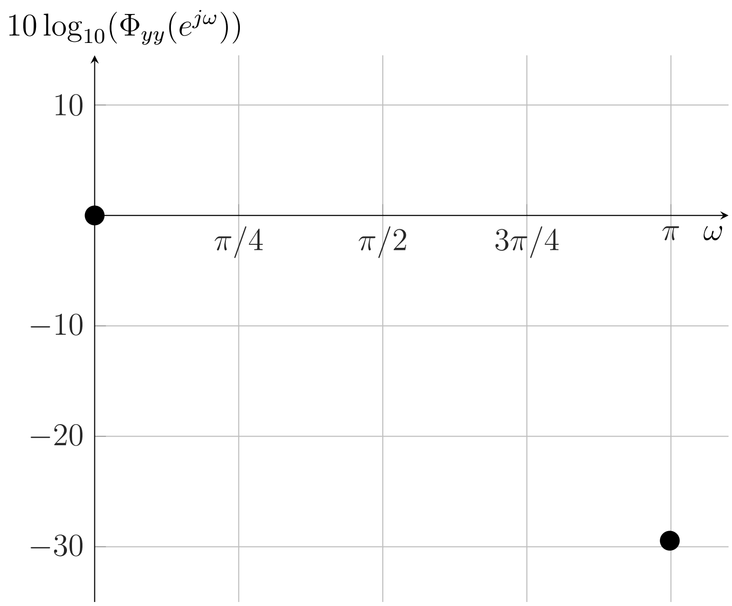


Figure 2: Sketch of the PSD of the output of $h[n]$ for a white noise input.

(d) (3 points) Give an expression for the average power of the output signal $y[n]$. Your answer should depend on σ_x^2 and either $c_{hh}[n]$ or $H(e^{j\omega})$.

(e) (8 points) Note from Figure 1a that $H(z)$ is not minimum phase. Sketch the pole-zero diagram of $H_{min}(z)$ and $H_{ap}(z)$ so that $H(z) = H_{min}(z)H_{ap}(z)$, where $H_{min}(z)$ is a minimum-phase system and $H_{ap}(z)$ is an all-pass system.

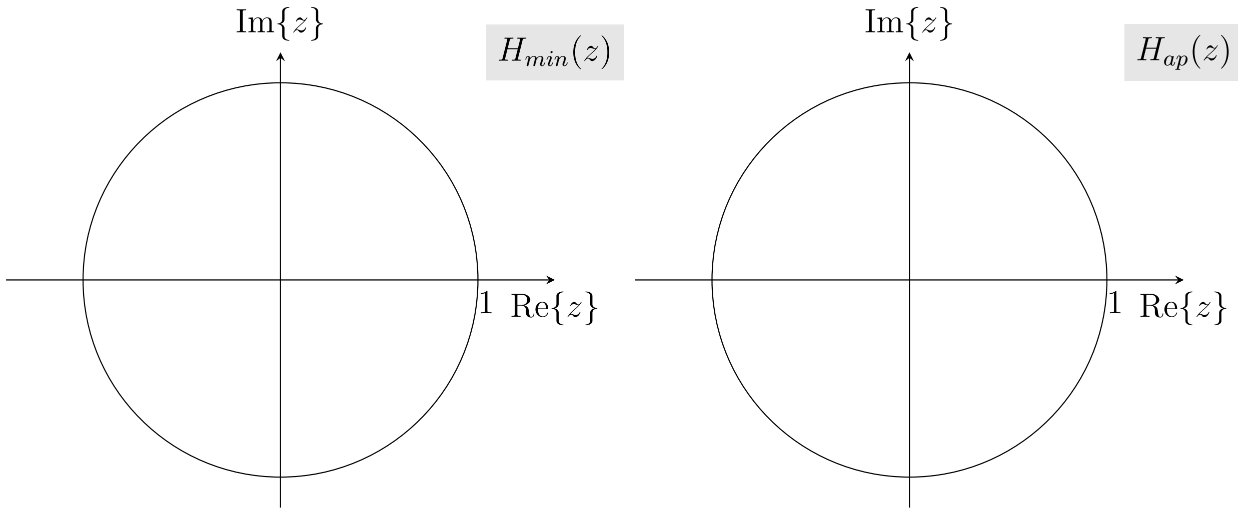


Figure 3: Minimum-phase and all-pass decomposition of $H(z)$.

Problem 2: (15 points)

Consider the generic discrete-time system discussed in class:

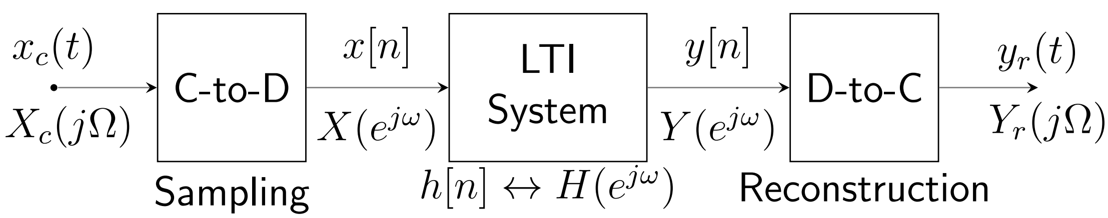


Figure 4: Diagram for discrete-time processing of continuous-time signals.

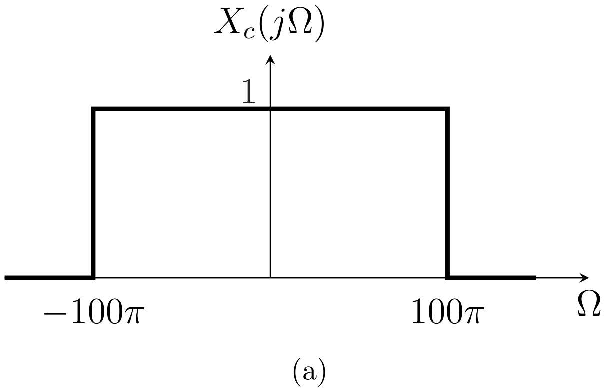
Let's assume that input signal $x_c(t)$ and the discrete-time LTI system $h[n]$ have the following frequency-domain representation:

We'll consider two different scenarios. In the first, sampling is performed at the Nyquist rate. That is, $\Omega_s = 200\pi$. In the second scenario, we'll assume that the signal is under-sampled with $\Omega_s = 160\pi$.

(a) For each scenario, sketch the Fourier transform of the discrete-time signal $x[n]$.

(b) For each scenario, sketch the Fourier transform of the output signal after the discrete-time LTI system.

(c) Suppose that for reconstruction, we use the ideal lowpass filter with cutoff frequency $\Omega_s/2$. For each case, sketch the Fourier transform of the output continuous-time signal $y_r(t)$.



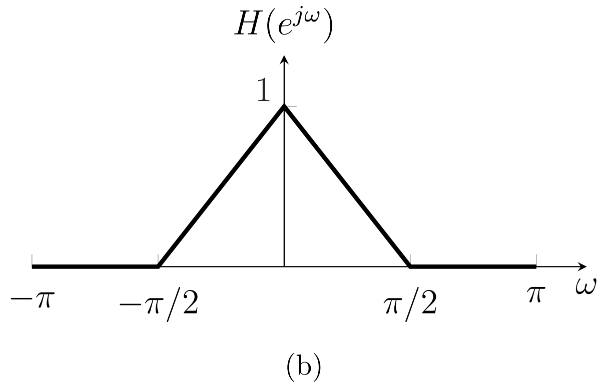


Figure 5: (a) Fourier transform of the input signal $x_c(t)$ and (b) frequency response of the discrete-time system.

Problem 3: (15 points) (from mid-term exam, fall 2009)

A continuous-time random signal $x_c(t)$ has autocorrelation function $\phi_{x_cx_c}(\tau)$ and corresponding power spectrum $\Phi_{x_cx_c}(j\Omega)$, which is shown in Figure 6.

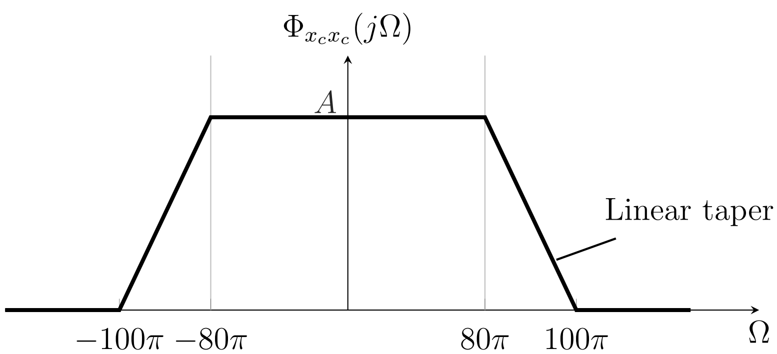


Figure 6: PSD of the continuous-time random signal.

This signal is sampled with sampling period T to yield the discrete-time random signal $x[n] = x_c(nT)$.

(a) Determine the sampling period T so that the power spectrum of the discrete-time sampled sequence is a constant, i.e., of the form

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \quad |\omega| \leq \pi.$$

In other words, choose T so that the resulting sampled signal is a discrete-time white noise signal.

(b) Determine the value of the average power σ_x^2.

(c) Is the result of part (a) only possible because of the linear taper of the continuous-time power spectrum? If not, can you think of a general condition such that a non-white continuous-time spectrum will become a white spectrum after sampling?
Hint: Consider what conditions the autocorrelation function would have to meet.

Problem 4: (15 points)

Consider the continuous-time representation of the process of sampling followed by reconstruction shown in Figure 7.

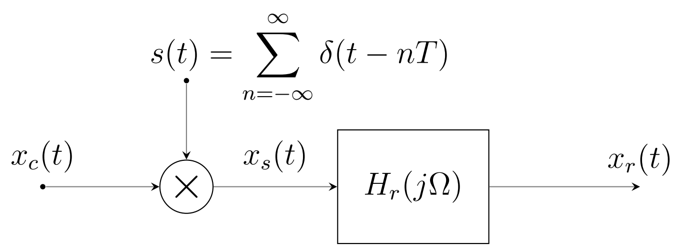


Figure 7: Representation of sampling and reconstruction.

Assume that the input signal is

$$x_c(t) = 15 + 10\cos(600\pi t) + 5\cos(1500\pi t - \pi/3) \qquad -\infty < t < \infty$$

The frequency response of the reconstruction filter is

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \leq \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}$$

(a) Determine the continuous-time Fourier transform $X_c(j\Omega)$ and plot it as a function of Ω.

(b) Assume that $f_s = 1/T = 2000$ samples/sec and plot the Fourier transform $X_s(j\Omega)$ as a function of Ω for $-2\pi/T \leq \Omega \leq 2\pi/T$. What is the output $x_r(t)$ in this case? (You should be able to give an exact equation for $x_r(t)$.

(c) Is it possible to choose the sampling rate so that

$$x_r(t) = A + B\cos(\Omega_0 t)$$

where A is a constant? If so, what is the required sampling rate $f_s = 1/T$, and what are the numerical values of A, B, and Ω_0?

Problem 5: (25 points)

Consider the two-channel microphone array depicted in Figure 8. The lowpass filtered microphone array signals are modeled by the following equations:

$$x_{c1}(t) = \alpha_1 s_c(t) + v_{c1}(t) \tag{2a}$$

$$x_{c2}(t) = \alpha_2 s_c(t + t_d) + v_{c2}(t), \tag{2b}$$

where $s_c(t)$ is the source component of the output of the lowpass filter of channel 1. (All signals and noises are real signals). The time-delay difference, t_d, between the two transmission paths depends on the source location, the velocity of sound (denoted c), and the

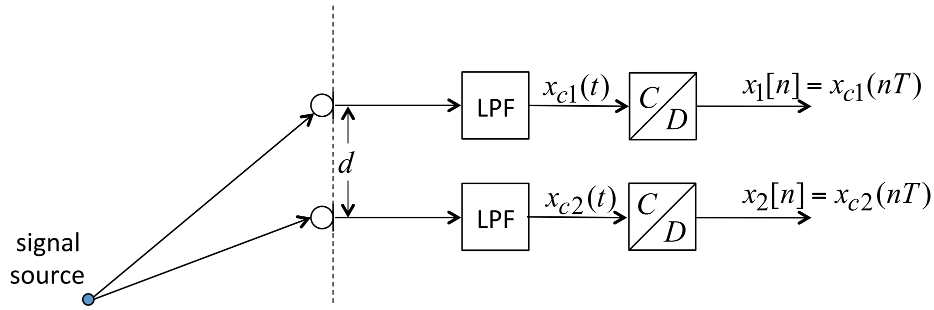


Figure 8: Two-channel microphone array.

microphone spacing, d. Because of the lowpass anti-aliasing filters (LPF blocks), the source signal $s_c(t)$ and the noise signals $v_{c1}(t)$ and $v_{c2}(t)$ are assumed to be bandlimited random signals whose power spectra have highest radian continuous-time frequency Ω_N. We also assume that all signals are stationary with zero mean. The two filtered microphone output signals are sampled with sampling rate $2\pi/T \geq 2\Omega_N$ to give the discrete-time signals

$$x_1[n] = \alpha_1 s[n] + v_1[n] \tag{3a}$$
$$x_2[n] = \alpha_2 s_D[n] + v_2[n], \tag{3b}$$

where $s[n] = s_c(nT)$ and $s_D[n] = s_c(nT + t_d)$. When $D = t_d/T$ is an integer, we may write $s_D[n] = s[n + D]$.

(a) Depending on the source location, t_d may be either positive or negative. What is the range of time difference values t_d that we can have if the sound source is somewhere (even just slightly) to the left of the dashed line through the microphones?

(b) Assuming that D is an integer, determine an expression for the cross-correlation function $\phi_{x_1x_2}[m] = \mathbb{E}\{x_1[n+m]x_2[n]\}$. You may assume that the signal and noises are statistically independent.

(c) Determine the cross-power spectrum $\Phi_{x_1x_2}(e^{j\omega})$, which is the DTFT of $\phi_{x_1x_2}[m]$.

(d) Where will the maximum value of $\phi_{x_1x_2}[m]$ occur? How could this result be used in an algorithm for estimating the time delay t_d?

(e) Assume that the sampled signal component $s[n]$ is a white random signal with power spectrum $\Phi_{ss}(e^{j\omega}) = \sigma_s^2$ for all ω. Using the result of (c) obtain an expression for the cross-power spectrum, $\Phi_{x_1x_2}(e^{j\omega})$, and use it to determine an expression for $\phi_{x_1x_2}[m]$ that is valid even when $D = t_d/T$ is *not* an integer. Specialize your result for the case when D is an integer, i.e., when t_d is an integer multiple of the sampling period T.