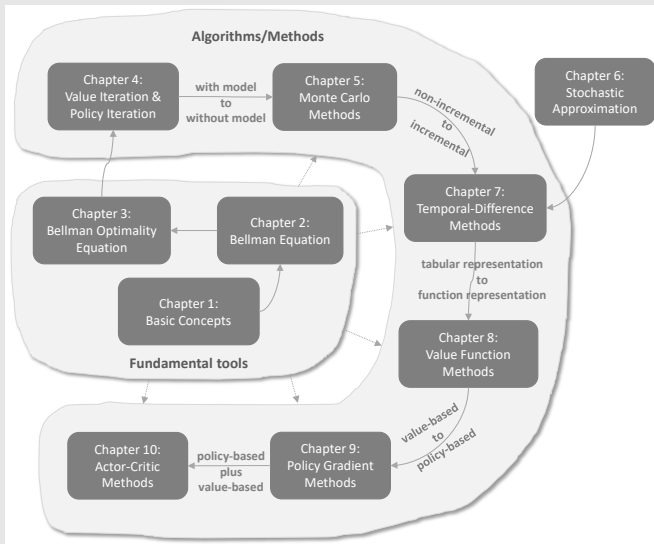


Lecture 8: Value Function Methods

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Westlake University

Outline



- 1 Motivating examples: from table to function
- 2 Algorithm for state value estimation
 - Objective function
 - Optimization algorithms
 - Selection of function approximators
 - Illustrative examples
 - Summary of the story
 - Theoretical analysis (optional)
- 3 Sarsa with function approximation
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Motivating examples: from table to function

So far in this book, state and action values are represented by [tables](#).

- For example, state value:

State	s_1	s_2	$\dots$	s_n
Value	$v_\pi(s_1)$	$v_\pi(s_2)$	$\dots$	$v_\pi(s_n)$

- For example, action value:

	a_1	a_2	a_3	a_4	a_5
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s_9	$q_\pi(s_9, a_1)$	$q_\pi(s_9, a_2)$	$q_\pi(s_9, a_3)$	$q_\pi(s_9, a_4)$	$q_\pi(s_9, a_5)$

- Advantage: intuitive and easy to analyze
- Disadvantage: difficult to handle large or continuous state or action spaces.
Two aspects: 1) storage; 2) generalization ability

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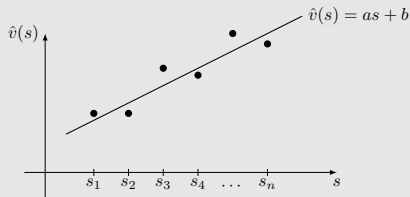
- **Advantage:** intuitive and easy to analyze
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Consider an example:

- There are n states: $s_1, \dots, s_n$.
- The state values are $v_\pi(s_1), \dots, v_\pi(s_n)$, where π is a given policy.
- n is very large!
- We hope to use a simple curve to approximate these values.

Motivating examples: from table to function

For example, we can use a simple **straight line** to fit the dots.



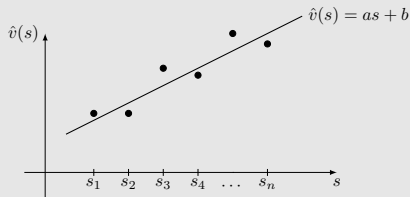
Suppose the equation of the straight line is

$$\hat{v}(s, w) = as + b = \underbrace{[s, 1]}_{\phi^T(s)} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_w = \phi^T(s)w$$

w is the parameter vector; $\phi(s)$ the feature vector of s ; $\hat{v}(s, w)$ is linear in w .

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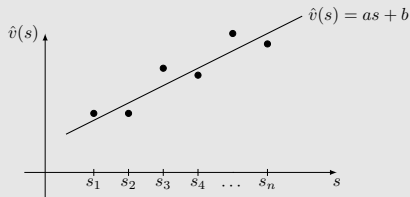
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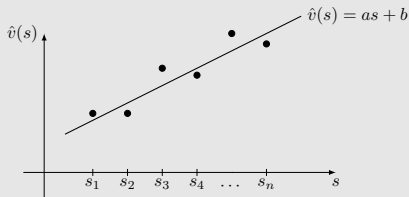
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Difference between the tabular and function methods:

Difference 1: How to retrieve the value of a state

- When the values are represented by a table, we can directly read the value in the table.
- When the values are represented by a function, we need to input the state index s into the function and calculate the function value.

For example, $s \rightarrow \phi(s) \rightarrow \phi^T(s)w = \hat{v}(s, w)$

- Benefit: storage. We do not need to store $|S|$ state values. We only need to store a lower-dimensional w .

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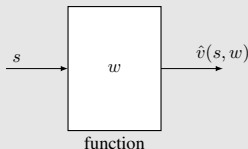
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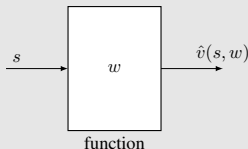
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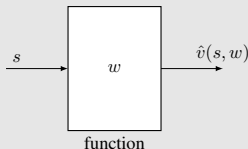
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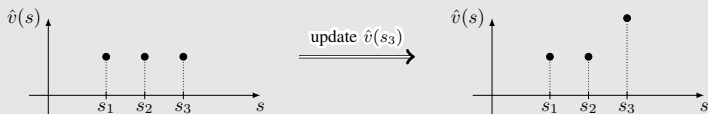
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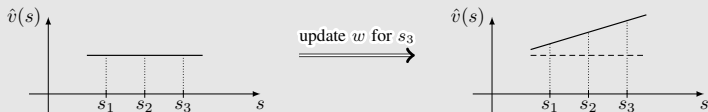
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Difference 2: How to update the value of a state



(a) Tabular method



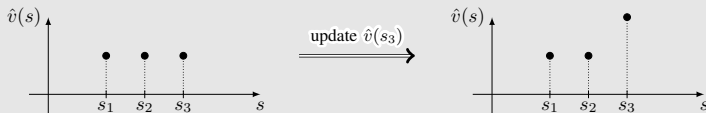
(b) Function method

Benefit: generalization ability. When we update $\hat{v}(s, w)$ by changing w , the values of the neighboring states are also changed.

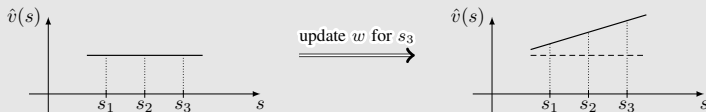
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Motivating examples: from table to function

The benefits are **not free**. It comes with a **cost**: the state values can not be represented accurately. This is why this method is called **approximation**.

We can fit the points more precisely using high-order curves:

$$\hat{v}(s, w) = as^2 + bs + c = \underbrace{[s^2, s, 1]}_{\phi^T(s)} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_w = \phi^T(s)w.$$

In this case,

- The dimensions of w and $\phi(s)$ increase; the values may be fitted more accurately.
- Although $\hat{v}(s, w)$ is nonlinear in s , it is linear in w . The nonlinearity is contained in $\phi(s)$.

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- **Idea:** Approximate the state and action values using **parameterized functions**: $\hat{v}(s, w) \approx v_{\pi}(s)$ where $w \in \mathbb{R}^m$ is the parameter vector.
- **Key difference:** How to retrieve and change the value of $v(s)$
- **Advantages:**
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Introduce in a more formal way:

- Let $v_\pi(s)$ and $\hat{v}(s, w)$ be the true state value and the estimated state value, respectively.
- Our goal is to find an optimal w so that $\hat{v}(s, w)$ can best approximate $v_\pi(s)$ for every s .
- This is a policy evaluation problem. Later we will extend to policy improvement.

To find the optimal w , we need two steps.

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$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))^2].$$

- Our goal is to find the best w that can minimize $J(w)$.
- The expectation is with respect to the random variable $S \in \mathcal{S}$.

What is the probability distribution of S ?

- This is new. We have not discussed the probability distribution of states so far.
- There are several ways to define the probability distribution of S .

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- The expectation is with respect to the random variable $S \in \mathcal{S}$.

What is the probability distribution of S ?

- This is new. We have not discussed the probability distribution of states so far.
- There are several ways to define the probability distribution of S .

The first way is to use a **uniform distribution**.

- That is to treat all the states to be equally important by setting the probability of each state as $1/|S|$.
- In this case, the objective function becomes

$$J(w) = \mathbb{E}[(v_{\pi}(S) - \hat{v}(S, w))^2] = \frac{1}{|S|} \sum_{s \in S} (v_{\pi}(s) - \hat{v}(s, w))^2.$$

- **Drawback:**
 - The states may not be equally important. For example, some states may be rarely visited by a policy. Hence, this way does not consider the real dynamics of the Markov process under the given policy.

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The second way is to use the stationary distribution.

- Stationary distribution is an important concept that will be frequently used in this course. It describes the long-run behavior of a Markov process.
- Let $\{d_\pi(s)\}_{s \in \mathcal{S}}$ denote the stationary distribution of the Markov process under policy π . By definition, $d_\pi(s) \geq 0$ and $\sum_{s \in \mathcal{S}} d_\pi(s) = 1$.
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This function is a weighted squared error.

- Since more frequently visited states have higher values of $d_\pi(s)$, their weights in the objective function are also higher than those rarely visited states.

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More explanation about stationary distribution:

- *Distribution*: Distribution of the state
- *Stationary*: Long-run behavior
- *Summary*: after the agent runs a long time following a policy, the probability that the agent is at any state can be described by this distribution.

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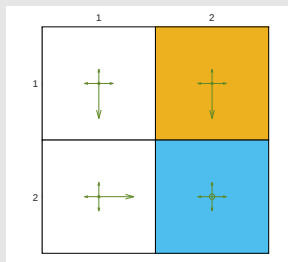
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Objective function - Stationary distribution

Illustrative example:

- Given a policy shown in the figure.
- Let $n_{\pi}(s)$ denote the number of times that s has been visited in a very long episode generated by π .
- Then, $d_{\pi}(s)$ can be approximated by

$$d_{\pi}(s) \approx \frac{n_{\pi}(s)}{\sum_{s' \in \mathcal{S}} n_{\pi}(s')}$$

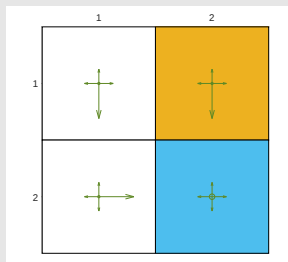


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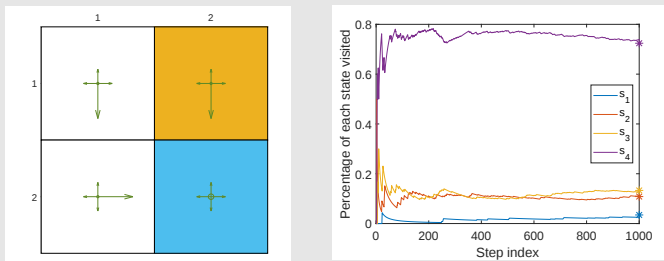


Figure: Long-run behavior of an ϵ -greedy policy with $\epsilon = 0.5$.

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The converged values can be predicted because they are the entries of d_π :

$$d_\pi^T = d_\pi^T P_\pi$$

For this example, we have P_π as

$$P_\pi = \begin{bmatrix} 0.3 & 0.1 & 0.6 & 0 \\ 0.1 & 0.3 & 0 & 0.6 \\ 0.1 & 0 & 0.3 & 0.6 \\ 0 & 0.1 & 0.1 & 0.8 \end{bmatrix}.$$

It can be calculated that the left eigenvector for the eigenvalue of one is

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While we have the objective function, the next step is to optimize it.

- To minimize the objective function $J(w)$, we can use the **gradient-descent** algorithm:

$$w_{k+1} = w_k - \alpha_k \nabla_w J(w_k)$$

The true gradient is

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The true gradient above involves the calculation of an expectation.

We can use the stochastic gradient to replace the true gradient:

$$w_{k+1} = w_k + \alpha_k \mathbb{E}[(v_\pi(S) - \hat{v}(S, w)) \nabla_w \hat{v}(S, w)]$$

$$\Downarrow$$

$$w_{t+1} = w_t + \alpha_t (v_\pi(s_t) - \hat{v}(s_t, w_t)) \nabla_w \hat{v}(s_t, w_t)$$

where s_t is a sample of S . Here, $2\alpha_t$ is merged to α_t .

- The samples are expected to satisfy the stationary distribution. In practice, they may not satisfy.
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In particular,

- **First, Monte Carlo learning with function approximation**

Let g_t be the discounted return starting from s_t in the episode. Then, g_t can be used to approximate $v_\pi(s_t)$. The algorithm becomes

$$w_{t+1} = w_t + \alpha_t (g_t - \hat{v}(s_t, w_t)) \nabla_w \hat{v}(s_t, w_t).$$

- **Second, TD learning with function approximation**

By the spirit of TD learning, $r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t)$ can be viewed as an approximation of $v_\pi(s_t)$. Then, the algorithm becomes

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Pseudocode: TD learning of state values with function approximation

Initialization: A function $\hat{v}(s, w)$ that is differentiable in w . Initial parameter w_0 .

Goal: Learn the true state values of a given policy π .

For each episode $\{(s_t, r_{t+1}, s_{t+1})\}_t$ generated by π , do

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An important question that has not been answered: **How to select the function $\hat{v}(s, w)$?**

- The first approach, which was **widely used before**, is to use a linear function

$$\hat{v}(s, w) = \phi^T(s)w$$

Here, $\phi(s)$ is the feature vector, which can be a polynomial basis, Fourier basis, ... (see my book for details). We have seen in the motivating example and will see again in the illustrative examples later.

- The second approach, which is **widely used nowadays**, is to use a neural network as a nonlinear function approximator.
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In the linear case where $\hat{v}(s, w) = \phi^T(s)w$, we have

$$\nabla_w \hat{v}(s, w) = \phi(s).$$

Substituting the gradient into the TD algorithm

$$w_{t+1} = w_t + \alpha_t [r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t) - \hat{v}(s_t, w_t)] \nabla_w \hat{v}(s_t, w_t)$$

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 - The theoretical properties of the TD algorithm in the linear case can be much better understood than in the nonlinear case.
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We next show that **tabular representation is a special case of linear function representation**. Hence, the tabular and function representations are **unified!**

- Consider a special feature vector for state s :

$$\phi(s) = e_s \in \mathbb{R}^{|S|},$$

where e_s is a vector with the s th entry as 1 and the others as 0.

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This is a vector equation that merely updates the s_t th entry of w_t .

- Multiplying $e_{s_t}^T$ on both sides of the equation gives

$$w_{t+1}(s_t) = w_t(s_t) + \alpha_t (r_{t+1} + \gamma w_t(s_{t+1}) - w_t(s_t)),$$

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- 2 Algorithm for state value estimation
 - Objective function
 - Optimization algorithms
 - Selection of function approximators
 - **Illustrative examples**
 - Summary of the story
 - Theoretical analysis (optional)
- 3 Sarsa with function approximation
- 4 Q-learning with function approximation
- 5 Deep Q-learning
- 6 Summary

Illustrative examples

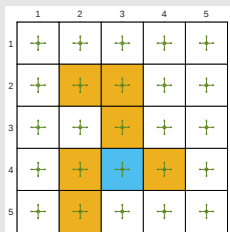
Consider a 5x5 grid-world example:

	1	2	3	4	5
1	+	+	+	+	+
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- Given a policy: $\pi(a|s) = 0.2$ for any s, a
- Our aim is to estimate the state values of this policy (policy evaluation problem).
- There are 25 state values in total. We next show that we can use less than 25 parameters to approximate 25 state values.
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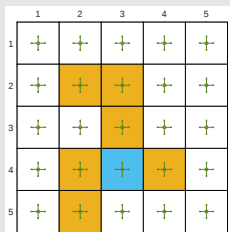
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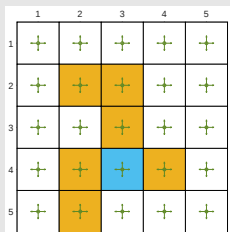
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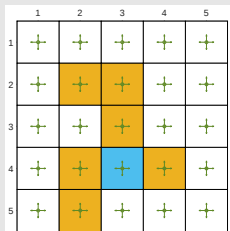
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Illustrative examples

Ground truth:

- The true state values and the 3D visualization

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4					
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	1	2	3	4	5
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Experience samples:

- 500 episodes were generated following the given policy.
- Each episode has 500 steps and starts from a randomly selected state-action pair following a uniform distribution.

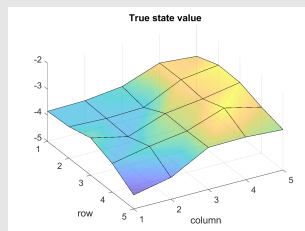
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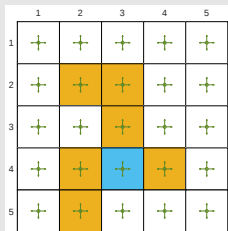
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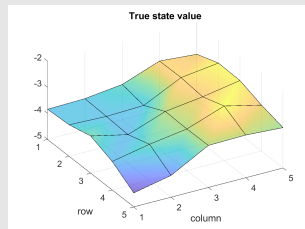
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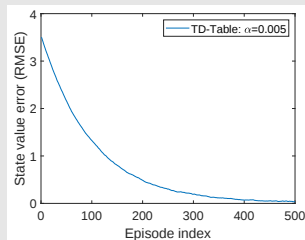
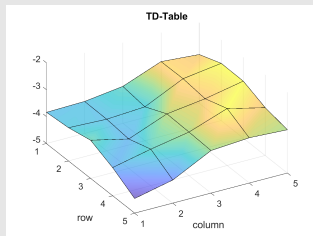


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TD-Table:

- For comparison, the results by the tabular TD algorithm (called **TD-Table** here):



TD-Linear:

- How to apply the TD-Linear algorithm?

- Feature vector selection:

$$\phi(s) = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \in \mathbb{R}^3.$$

- In this case, the approximated state value is

$$\hat{v}(s, w) = \phi^T(s)w = [1, x, y] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = w_1 + w_2x + w_3y.$$

Remark: $\phi(s)$ can also be defined as $\phi(s) = [x, y, 1]^T$, where the order of the elements does not matter.

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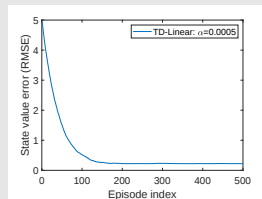
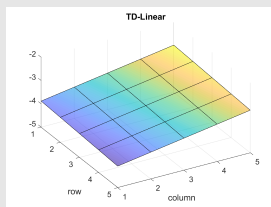
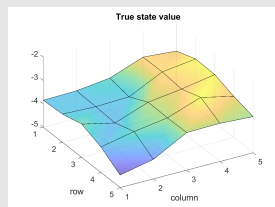
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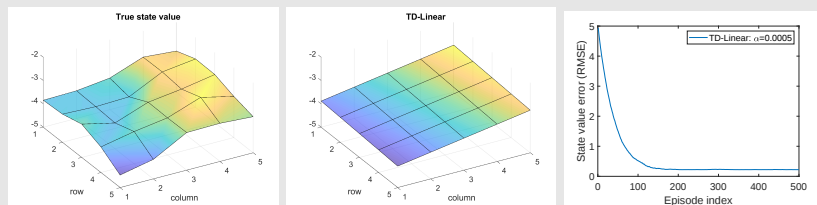


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- We are trying to use a plane to approximate a non-plane surface!

To enhance the approximation ability, we can use **high-order feature vectors** and hence **more parameters**.

- For example, we can consider

$$\phi(s) = [1, x, y, x^2, y^2, xy]^T \in \mathbb{R}^6.$$

In this case,

$$\hat{v}(s, w) = \phi^T(s)w = w_1 + w_2x + w_3y + w_4x^2 + w_5y^2 + w_6xy$$

which corresponds to a quadratic surface.

- We can further increase the dimension of the feature vector:

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$$\phi(s) = [1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2]^T \in \mathbb{R}^{10}.$$

To enhance the approximation ability, we can use **high-order feature vectors** and hence **more parameters**.

- For example, we can consider

$$\phi(s) = [1, x, y, x^2, y^2, xy]^T \in \mathbb{R}^6.$$

In this case,

$$\hat{v}(s, w) = \phi^T(s)w = w_1 + w_2x + w_3y + w_4x^2 + w_5y^2 + w_6xy$$

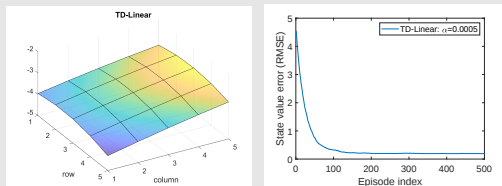
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Illustrative examples

Results by the TD-Linear algorithm with higher-order feature vectors:

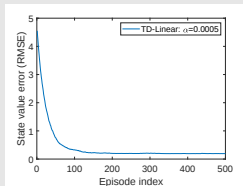
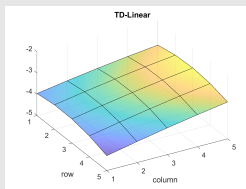


The above figure: $\phi(s) \in \mathbb{R}^6$

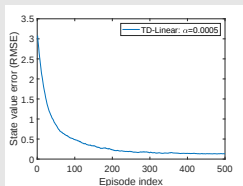
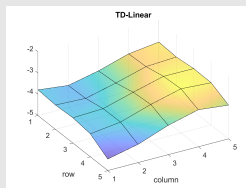
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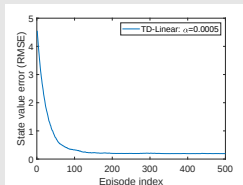
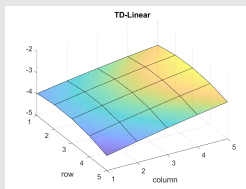
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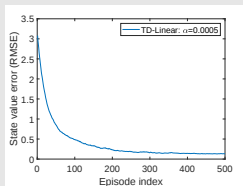
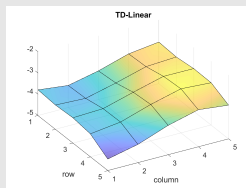
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- Illustrative examples
- **Summary of the story**
- Theoretical analysis (optional)

3 Sarsa with function approximation

4 Q-learning with function approximation

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6 Summary

Summary of the story

Up to now, we finished the story of TD learning with value function approximation.

- 1) This story started from the objective function:

$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))^2]$$

The objective function suggests that it is a policy evaluation problem.

- 2) The gradient-descent algorithm is

$$w_{t+1} = w_t + \alpha_t (v_\pi(s_t) - \hat{v}(s_t, w_t)) \nabla_w \hat{v}(s_t, w_t)$$

- 3) The true value function, which is unknown, in the algorithm is replaced by an approximation, leading to the algorithm:

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Although this story is very helpful to understand the basic idea, it is not mathematically rigorous.

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does not minimize the following objective function:

$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))^2]$$

Theoretical analysis (optional)

Different objective functions:

- **Objective function 1: True value error**

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- **Objective function 3: Projected Bellman error**

$$J_{PBE}(w) = \|\hat{v}(w) - MT_\pi(\hat{v}(w))\|_D^2,$$

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So far, we merely considered **state value estimation**. That is

$$\hat{v}(s) \approx v_{\pi}(s), \quad s \in \mathcal{S}$$

To search for optimal policies, we need to estimate action values.

The Sarsa algorithm with value function approximation is

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \hat{q}(s_{t+1}, a_{t+1}, w_t) - \hat{q}(s_t, a_t, w_t) \right] \nabla_w \hat{q}(s_t, a_t, w_t).$$

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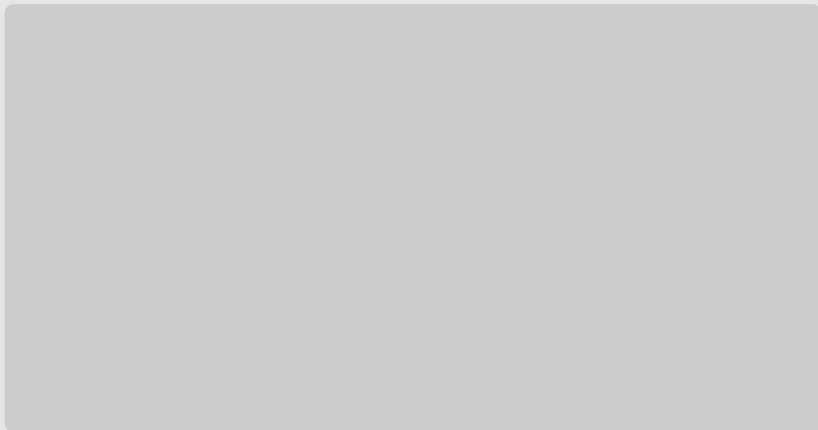
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Sarsa with function approximation

To search for optimal policies, we can combine **policy evaluation** and **policy improvement**.



Sarsa with function approximation

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Pseudocode: Sarsa with function approximation

Initialization: Initial parameter w_0 . Initial policy π_0 . $\alpha_t = \alpha > 0$ for all t . $\epsilon \in (0, 1)$.

Goal: Learn an optimal policy to lead the agent to the target state from an initial state s_0 .

For each episode, do

 Generate a_0 at s_0 following $\pi_0(s_0)$

 If s_t ($t = 0, 1, 2, \dots$) is not the target state, do

 Collect the experience sample $(r_{t+1}, s_{t+1}, a_{t+1})$ given (s_t, a_t) : generate r_{t+1}, s_{t+1} by interacting with the environment; generate a_{t+1} following $\pi_t(s_{t+1})$.

Update q-value (update parameter):

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \hat{q}(s_{t+1}, a_{t+1}, w_t) - \hat{q}(s_t, a_t, w_t) \right] \nabla_w \hat{q}(s_t, a_t, w_t)$$

Update policy:

$$\pi_{t+1}(a|s_t) = 1 - \frac{\epsilon}{|\mathcal{A}(s_t)|} (|\mathcal{A}(s_t)| - 1) \text{ if } a = \arg \max_{a \in \mathcal{A}(s_t)} \hat{q}(s_t, a, w_{t+1})$$

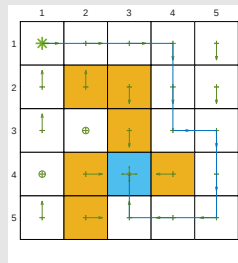
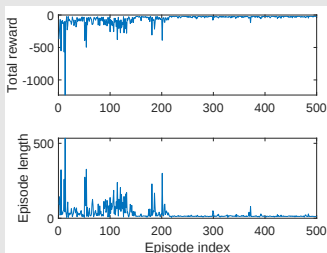
$$\pi_{t+1}(a|s_t) = \frac{\epsilon}{|\mathcal{A}(s_t)|} \text{ otherwise}$$

$$s_t \leftarrow s_{t+1}, a_t \leftarrow a_{t+1}$$

Sarsa with function approximation

Illustrative example:

- Sarsa with **linear function** approximation: $\hat{q}(s, a, w) = \phi^T(s, a)w$
- $\gamma = 0.9$, $\epsilon = 0.1$, $r_{\text{boundary}} = r_{\text{forbidden}} = -10$, $r_{\text{target}} = 1$, $\alpha = 0.001$.



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Similar to Sarsa, tabular Q-learning can also be extended to the case of value function approximation.

The q-value update rule is

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \max_{a \in \mathcal{A}(s_{t+1})} \hat{q}(s_{t+1}, a, w_t) - \hat{q}(s_t, a_t, w_t) \right] \nabla_w \hat{q}(s_t, a_t, w_t),$$

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Pseudocode: Q-learning with function approximation (on-policy version)

Initialization: Initial parameter w_0 . Initial policy π_0 . $\alpha_t = \alpha > 0$ for all t . $\epsilon \in (0, 1)$.

Goal: Learn an optimal path to lead the agent to the target state from an initial state s_0 .

For each episode, do

 If s_t ($t = 0, 1, 2, \dots$) is not the target state, do

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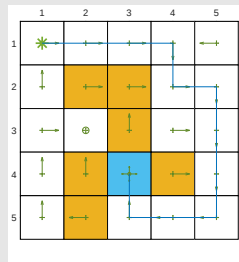
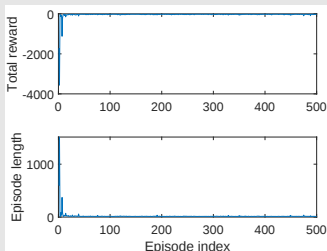
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Q-learning with function approximation

Illustrative example:

- Q-learning with **linear function** approximation: $\hat{q}(s, a, w) = \phi^T(s, a)w$
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Deep Q-learning or deep Q-network (DQN):

- One of the earliest and most successful algorithms that introduce deep neural networks into RL.
- The role of neural networks is to be a nonlinear function approximator.
- Different from the following algorithm:

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$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \max_{a \in \mathcal{A}(s_{t+1})} \hat{q}(s_{t+1}, a, w_t) - \hat{q}(s_t, a_t, w_t) \right] \nabla_w \hat{q}(s_t, a_t, w_t)$$

because of the way of training a network.

Deep Q-learning aims to minimize the objective function/loss function:

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \max_{a \in \mathcal{A}(s_{t+1})} \hat{q}(s_{t+1}, a, w_t) - \hat{q}(s_t, a_t, w_t) \right] \nabla_w \hat{q}(s_t, a_t, w_t)$$

$\Downarrow$

$$J(w) = \mathbb{E} \left[\left(R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w) - \hat{q}(S, A, w) \right)^2 \right]$$

where (S, A, R, S') are random variables.

How to minimize the objective function? Gradient-descent!

- How to calculate the gradient of the objective function? Tricky!
- That is because, in this objective function

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the parameter w not only appears in $\hat{q}(S, A, w)$ but also in

$$y \doteq R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w)$$

- Since the optimal a depends on w ,

$$\nabla_w y \neq \gamma \max_{a \in \mathcal{A}(S')} \nabla_w \hat{q}(S', a, w)$$

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Technique 1: Two networks, a main network and a target network.

Why is it used?

- The mathematical reason has been explained when we calculate the gradient.

Implementation details:

- Let w and w_T denote the parameters of the main and target networks, respectively. They are set to be the same initially.
- In every iteration, we draw a mini-batch of samples $\{(s, a, r, s')\}$ from the replay buffer (will be explained later).
- For every (s, a, r, s') , we can calculate the desired output as

$$y_T \doteq r + \gamma \max_{a \in \mathcal{A}(s')} \hat{q}(s', a, w_T)$$

Therefore, we obtain a mini-batch of data:

$$\{(s, a, y_T)\}$$

- Use $\{(s, a, y_T)\}$ to train the network so as to minimize $(y_T - \hat{q}(s, a, w))^2$.

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Technique 2: Experience replay

Question: What is experience replay?

Answer:

- After we have collected some experience samples, we do NOT use these samples in the order they were collected.
- Instead, we store them in a set, called replay buffer $\mathcal{B} \doteq \{(s, a, r, s')\}$
- Every time we train the neural network, we can draw a mini-batch of random samples from the replay buffer.
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Question: Why is experience replay necessary in deep Q-learning? Why does the replay must follow a uniform distribution?

Answer: The answers lie in the objective function.

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- $R \sim p(R|S, A)$, $S' \sim p(S'|S, A)$: R and S are determined by the system model.
- $(S, A) \sim d$: (S, A) is an index and treated as a single random variable
- The distribution of the state-action pair (S, A) is assumed to be uniform.
 - Why uniform distribution? Because no prior knowledge.
 - Can we use stationary distribution like before? No, since no policy is given.

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Revisit the tabular case:

- Question: Why does not tabular Q-learning require experience replay?
 - Answer: Because it does not require any distribution of S or A .
- Question: Why does Deep Q-learning involve distributions?
 - Answer: Because we need to define a *scalar* objective function $J(w) = \mathbb{E}[*]$, where $\mathbb{E}$ is for all (S, A) .
 - The tabular case aims to solve a set of equations for all (s, a) (Bellman optimality equation), whereas the deep case aims to optimize a scalar objective function.
- Question: Can we use experience replay in tabular Q-learning?
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- Question: Why does not tabular Q-learning require experience replay?
 - Answer: Because it does not require any distribution of S or A .
- Question: Why does Deep Q-learning involve distributions?
 - Answer: Because we need to define a *scalar* objective function $J(w) = \mathbb{E}[*]$, where $\mathbb{E}$ is for all (S, A) .
 - The tabular case aims to solve a set of equations for all (s, a) (Bellman optimality equation), whereas the deep case aims to optimize a scalar objective function.
- Question: Can we use experience replay in tabular Q-learning?
 - Answer: Yes, we can. And more sample efficient (why?)

Pseudocode: Deep Q-learning (off-policy version)

Initialization: A main network and a target network with the same initial parameter.

Goal: Learn an optimal target network to approximate the *optimal* action values from the experience samples generated by a given behavior policy π_b .

Store the experience samples generated by π_b in a replay buffer $\mathcal{B} = \{(s, a, r, s')\}$

For each iteration, do

Uniformly draw a mini-batch of samples from $\mathcal{B}$

For each sample (s, a, r, s') , calculate the target value as $y_T = r + \gamma \max_{a \in \mathcal{A}(s')} \hat{q}(s', a, w_T)$, where w_T is the parameter of the target network

Update the main network to minimize $(y_T - \hat{q}(s, a, w))^2$ using the mini-batch of samples

Set $w_T = w$ every C iterations

Remarks:

- Why no policy update?
- The network input and output are different from the DQN paper.

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Illustrative example:

- We need to learn optimal action values for every state-action pair.
- Once the optimal action values are obtained, the optimal greedy policy can be obtained immediately.

Setup:

- One single episode is used to train the network.
- This episode is generated by an exploratory behavior policy shown in Fig. (a).
- The episode only has 1,000 steps! The tabular Q-learning requires 100,000 steps.
- A shallow neural network with one single hidden layer is used as a nonlinear approximator of $\hat{q}(s, a, w)$. The hidden layer has 100 neurons.

See details in the book.

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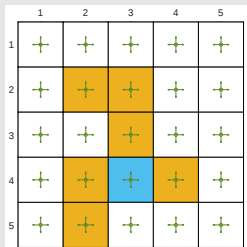
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Deep Q-learning



The behavior policy.

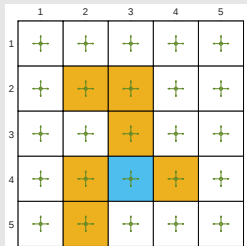
An episode of 1,000 steps.

The obtained policy.

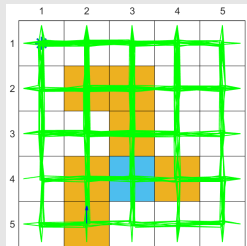
The TD error converges to zero.

The state estimation error converges to zero.

Deep Q-learning



The behavior policy.



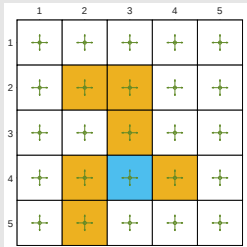
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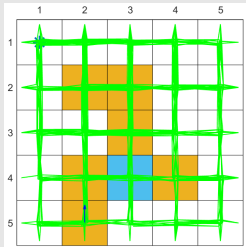
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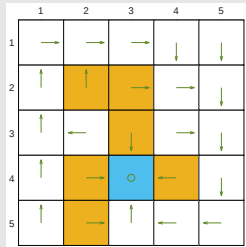
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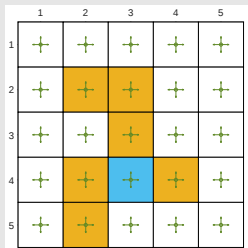


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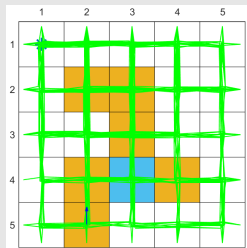
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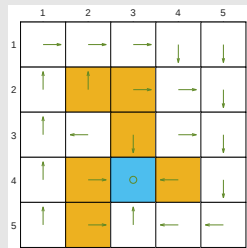
Deep Q-learning



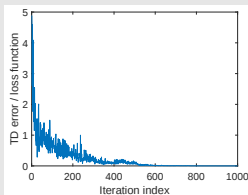
The behavior policy.



An episode of 1,000 steps.



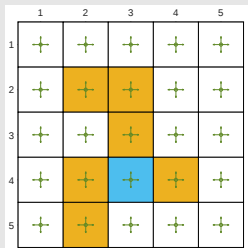
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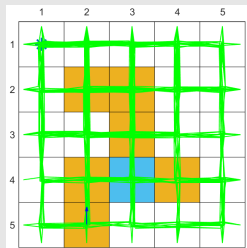
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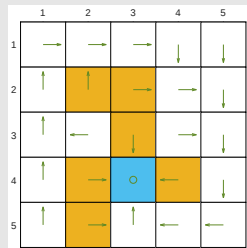
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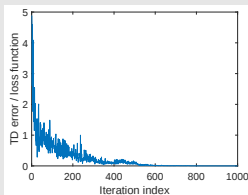
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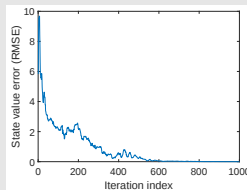
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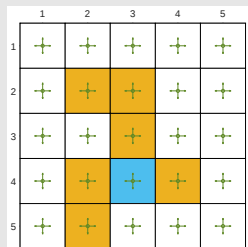


The TD error converges to zero.



The state estimation error converges to zero.

What if we only use a single episode of 100 steps? Insufficient data



The behavior policy.

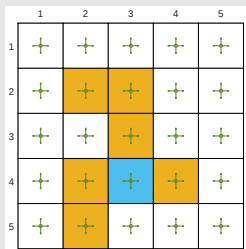
An episode of 100 steps.

The final policy.

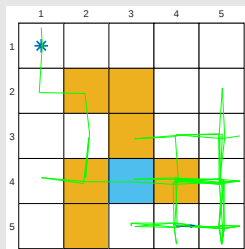
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The state error does not converge to zero.

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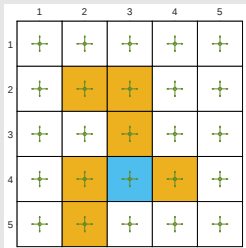
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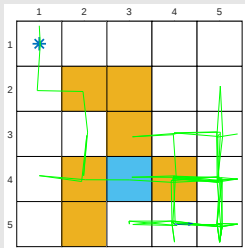
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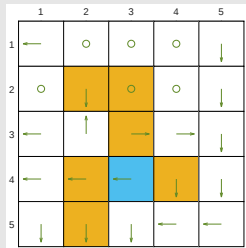
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An episode of 100 steps.

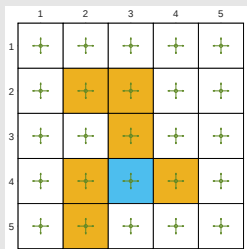


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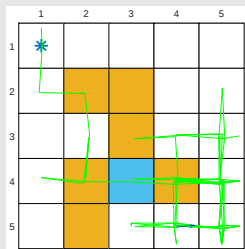
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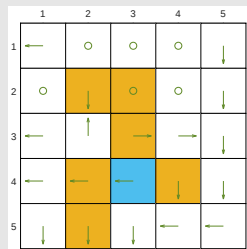
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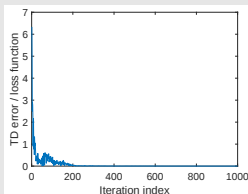
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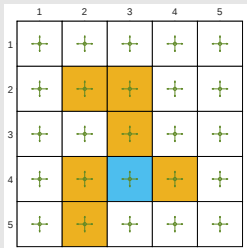
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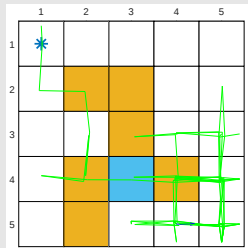
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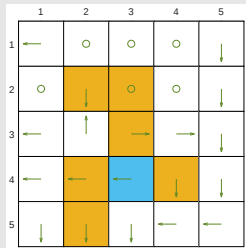
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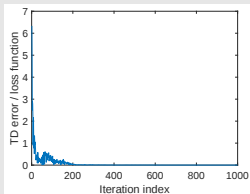
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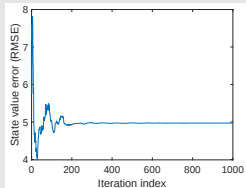
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- 1 Motivating examples: from table to function
- 2 Algorithm for state value estimation
 - Objective function
 - Optimization algorithms
 - Selection of function approximators
 - Illustrative examples
 - Summary of the story
 - Theoretical analysis (optional)
- 3 Sarsa with function approximation
- 4 Q-learning with function approximation
- 5 Deep Q-learning
- 6 Summary**

This lecture introduces the method of value function approximation.

- First, understand the basic idea.
- Second, understand the basic algorithms.